Weak convergence rates for spatial spectral Galerkin approximations of semilinear stochastic wave equations with multiplicative noise

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joint works with
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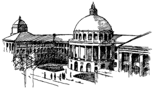


Throughout this talk consider

- $T \in (0, \infty)$, stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$,
- non-trivial separable Hilbert spaces H and U,
- id_U-cylindrical $(\mathcal{F}_t)_{t\in[0,T]}$-Wiener process $(W_t)_{t\in[0,T]}$.

Stochastic wave equations

Consider the case $H = U = L^2((0,1);\mathbb{R})$ and

- $f, b \in C_{\mathrm{b}}^{\infty}(\mathbb{R}, \mathbb{R})$,
- $\xi = (\xi_0, \xi_1) \in H_0^1((0,1);\mathbb{R}) \times H$,
- a mild solution $X\colon [0,T] \times \Omega \to H$ of

$$\ddot{X}_t(x) = \tfrac{\partial^2}{\partial x^2} X_t(x) + f(X_t(x)) + b(X_t(x))\, \dot{W}_t(x)$$

with $X_0(x) = \xi_0(x)$ and $\dot{X}_0(x) = \xi_1(x)$ for $t \in [0,T]$, $x \in (0,1)$.

The hyperbolic Anderson model

Consider the case $H = U = L^2((0,1);\mathbb{R})$ and

- $f, b \in C_{\mathrm{b}}^{\infty}(\mathbb{R},\mathbb{R})$ satisfying $f = 0$ and $b(x) = x$ for all $x \in \mathbb{R}$,
- $\xi = (\xi_0, \xi_1) \in H_0^1((0,1);\mathbb{R}) \times H$,
- a mild solution $X\colon [0,T] \times \Omega \to H$ of

$$\ddot{X}_t(x) = \tfrac{\partial^2}{\partial x^2} X_t(x) + X_t(x)\,\dot{W}_t(x)$$

with $X_0(x) = \xi_0(x)$ and $\dot{X}_0(x) = \xi_1(x)$ for $t \in [0,T]$, $x \in (0,1)$.

Abstract formulation

Consider

- an orthonormal basis $\{e_n\}_{n\in\mathbb{N}} \subseteq H$ of H,
- an increasing sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq (0,\infty)$,
- the linear operator $A\colon D(A) \subseteq H \to H$ which satisfies that $D(A) = \left\{v \in H\colon \sum_{n\in\mathbb{N}} |\lambda_n \langle e_n, v\rangle_H|^2 < \infty\right\}$ and

$$\forall v \in D(A)\colon \quad Av = \sum_{n\in\mathbb{N}} -\lambda_n \langle e_n, v\rangle_H e_n,$$

- a family of interpolation spaces H_r, $r \in \mathbb{R}$, associated to $-A$,
- the family of Hilbert spaces $\mathbf{H}_r = H_{r/2} \times H_{r/2-1/2}$, $r \in \mathbb{R}$,
- the linear operator $\mathbf{A}\colon D(\mathbf{A}) \subseteq \mathbf{H}_0 \to \mathbf{H}_0$ which satisfies that $D(\mathbf{A}) = \mathbf{H}_1$ and $\forall (v,w) \in \mathbf{H}_1\colon \quad \mathbf{A}(v,w) = (w, Av)$,
- $\xi \in \mathbf{H}_0$, $\mathbf{F} \in \operatorname{Lip}^0(\mathbf{H}_0, \mathbf{H}_0)$, $\mathbf{B} \in \operatorname{Lip}^0(\mathbf{H}_0, L_2(U, \mathbf{H}_0))$,
- a mild solution $\mathbf{X} = (X, \mathcal{X})\colon [0,T] \times \Omega \to \mathbf{H}_0$ of

$$\mathrm{d}\mathbf{X}_t = [\mathbf{A}\mathbf{X}_t + \mathbf{F}(\mathbf{X}_t)]\,\mathrm{d}t + \mathbf{B}(\mathbf{X}_t)\,\mathrm{d}W_t, \quad t \in [0,T],\ \mathbf{X}_0 = \xi.$$

Strong and weak convergence

Recall

- $\mathbf{X} = (X, \mathcal{X})\colon [0,T] \times \Omega \to \mathbf{H}_0$ is a mild solution of

$$\mathrm{d}\mathbf{X}_t = [\mathbf{A}\mathbf{X}_t + \mathbf{F}(\mathbf{X}_t)]\,\mathrm{d}t + \mathbf{B}(\mathbf{X}_t)\,\mathrm{d}W_t, \quad t \in [0,T],\ \mathbf{X}_0 = \xi$$

and consider

- approximation processes $\mathbf{X}^N\colon [0,T] \times \Omega \to \mathbf{H}_0$, $N \in \mathbb{N}$, of $\mathbf{X}$.

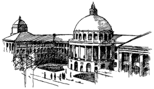
Strong convergence rates

$$\big(\mathbb{E}\big[\|\mathbf{X}_T - \mathbf{X}_T^N\|_{\mathbf{H}_0}^2\big]\big)^{1/2}, \quad N \in \mathbb{N},$$

are well understood. **Weak convergence rates**

$$\big|\mathbb{E}\big[\varphi(\mathbf{X}_T)\big] - \mathbb{E}\big[\varphi(\mathbf{X}_T^N)\big]\big|, \quad N \in \mathbb{N}, \quad \varphi \in C_{\mathrm{b}}^2(\mathbf{H}_0, \mathbb{R}),$$

have been investigated for about 12 years and are far away from being well understood.



Weak convergence results in the literature

- Hausenblas 2010 J. Comput. Appl. Math.
- Kovács, Lindner & Schilling 2014 arXiv
- Kovács, Larsson & Lindgren 2012 BIT
- Kovács, Larsson & Lindgren 2013 BIT
- Wang 2015 J. Sci. Comput.

All of the above mentioned references assume that $\mathbf{B}$ is constant.

Main setting

Recall

- $\mathbf{H}_r = H_{r/2} \times H_{r/2-1/2}$, $r \in \mathbb{R}$,
- $\xi \in \mathbf{H}_0$, $\mathbf{F} \in \mathrm{Lip}^0(\mathbf{H}_0, \mathbf{H}_0)$, $\mathbf{B} \in \mathrm{Lip}^0(\mathbf{H}_0, L_2(U, \mathbf{H}_0))$,

and consider

- $\{\mathbf{P}_N\}_{N \in \mathbb{N} \cup \{\infty\}} \subseteq L(\mathbf{H}_0)$ with $\forall (v, w) \in \mathbf{H}_0 \colon$ $\mathbf{P}_N(v, w) = \big(\sum_{n=1}^N \langle e_n, v \rangle_H e_n, \sum_{n=1}^N \langle e_n, w \rangle_H e_n\big)$,
- mild solutions $\mathbf{X}^N \colon [0, T] \times \Omega \to \mathbf{P}_N(\mathbf{H}_0)$ of

$$\mathrm{d}\mathbf{X}_t^N = [\mathbf{P}_N \mathbf{A} \mathbf{X}_t^N + \mathbf{P}_N \mathbf{F}(\mathbf{X}_t^N)]\, \mathrm{d}t + \mathbf{P}_N \mathbf{B}(\mathbf{X}_t^N)\, \mathrm{d}W_t$$

with $\mathbf{X}_0^N = \mathbf{P}_N(\xi)$ for $t \in [0, T]$, $N \in \mathbb{N} \cup \{\infty\}$.

Main result

Theorem (Jacobe de Naurois, Jentzen & W 2015)

Let $0 < \gamma$, $\gamma/2 < \beta \leq \gamma$, $0 \leq \rho \leq \gamma$ *and assume that*
$(-A)^{-\beta} \in L_1(H_0)$, $\xi \in \mathbf{H}_\gamma$, $\mathbf{F}|_{\mathbf{H}_\rho} \in \operatorname{Lip}^0(\mathbf{H}_\rho, \mathbf{H}_\gamma)$,
$\mathbf{B}|_{\mathbf{H}_\rho} \in \operatorname{Lip}^0(\mathbf{H}_\rho, L_2(U, \mathbf{H}_\rho) \cap L(U, \mathbf{H}_\gamma))$,
$\mathbf{F}|_{\bigcap_{r\in\mathbb{R}} \mathbf{H}_r} \in C_{\mathrm{b}}^2(\bigcap_{r\in\mathbb{R}} \mathbf{H}_r, \mathbf{H}_0)$,
$\mathbf{B}|_{\bigcap_{r\in\mathbb{R}} \mathbf{H}_r} \in C_{\mathrm{b}}^2(\bigcap_{r\in\mathbb{R}} \mathbf{H}_r, L_2(U, \mathbf{H}_0))$, *and*

$$\sup_{\substack{x\in\bigcap_{r\in\mathbb{R}} \mathbf{H}_r,\\ v_1,v_2\in\bigcap_{r\in\mathbb{R}} \mathbf{H}_r\setminus\{0\}}} \frac{\|\mathbf{F}''(x)(v_1, v_2)\|_{\mathbf{H}_0} + \|\mathbf{B}''(x)(v_1, v_2)\|_{L_2(U,\mathbf{H}_0)}}{\|v_1\|_{\mathbf{H}_0}\|v_2\|_{\mathbf{H}_0}} < \infty.$$

Then $\exists\, C \geq 0\colon \forall\, N \in \mathbb{N}$, $\forall\, \varphi \in C_{\mathrm{b}}^2(\mathbf{H}_0, \mathbb{R})\colon$

$$\left|\mathbb{E}\big[\varphi(\mathbf{X}_T^\infty)\big] - \mathbb{E}\big[\varphi(\mathbf{X}_T^N)\big]\right| \leq C \cdot (\lambda_N)^{\beta-\gamma}.$$

Comments on the main result

Proof uses

- the Kolmogorov equation,
- the Hölder inequality for Schatten norms,
- and the mild Itô formula.

Essentially sharp rate:

- There exist U, A, $\mathbf{F}$, $\mathbf{B}$, φ: $\forall \varepsilon > 0 \colon \exists C > 0 \colon \forall N \in \mathbb{N} \colon$

$$\big|\mathbb{E}\big[\varphi(\mathbf{X}_T^\infty)\big] - \mathbb{E}\big[\varphi(\mathbf{X}_T^N)\big]\big| \geq C \cdot (\lambda_N)^{\beta-\gamma-\varepsilon}.$$

Corollary (Hyperbolic Anderson model; Jacobe de Naurois, Jentzen & W 2015)

Let $H = L^2((0,1);\mathbb{R})$, $\varphi \in C^2_{\mathrm{b}}(H,\mathbb{R})$, $\xi = (\xi_0, \xi_1) \in H^1_0((0,1);\mathbb{R}) \times H$, *for every* $n \in \mathbb{N}$ *let* $e_n(\cdot) = \sqrt{2}\sin(n\pi(\cdot)) \in H$ *and for every* $N \in \mathbb{N} \cup \{\infty\}$ *let* $P_N(\cdot) = \sum_{n=1}^{N} \langle e_n, \cdot \rangle_H \, e_n \in L(H)$ *and let* $X^N \colon [0,T] \times \Omega \to P_N(H)$ *be a mild solution of*

$$\ddot{X}_t(x) = \tfrac{\partial^2}{\partial x^2} X_t(x) + P_N \, X_t(x) \, \dot{W}_t(x)$$

with $X_0(x) = (P_N \xi_0)(x)$ *and* $\dot{X}_0(x) = (P_N \xi_1)(x)$ *for* $t \in [0,T]$, $x \in (0,1)$. *Then* $\forall \, \varepsilon > 0 \colon \exists \, C \geq 0 \colon \forall \, N \in \mathbb{N} \colon$

$$\big| \mathbb{E}\big[\varphi(X_T^\infty)\big] - \mathbb{E}\big[\varphi(X_T^N)\big] \big| \leq C \cdot N^{\varepsilon - 1}.$$