Convolution quadrature for time-dependent Maxwell equations

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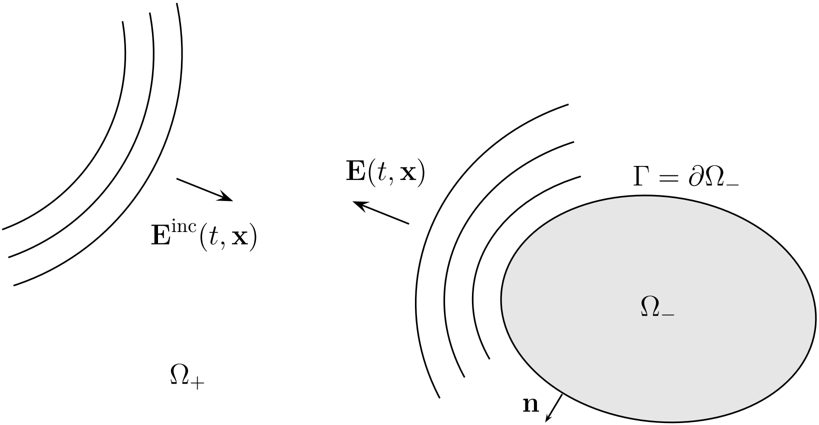
August 2009

Electromagnetic scattering problems



Maxwell's equations

Faraday's law of induction	$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{curl}\,\mathbf{E} = \mathbf{0}$
Gauss' law	$\operatorname{div} \mathbf{D} = \rho$
Ampere's circuital law	$-\frac{\partial \mathbf{D}}{\partial t} + \mathbf{curl}\,\mathbf{H} = \mathbf{J}$
Gauss' law for magnetism	$\operatorname{div} \mathbf{B} = 0.$

where

$\mathbf{E}$:	electric field	$\mathbf{B}$:	magnetic field
$\mathbf{D}$:	electric displacement	$\mathbf{H}$:	magnetic induction
$\mathbf{J}$:	current density	ρ:	charge density

Time-harmonic Maxwell equations

If we suppose the fields to be of the form

$$\mathbf{E}(\mathbf{x}, t) = \mathrm{Re}\left\{ \hat{\mathbf{E}}(\mathbf{x}) e^{-i\omega t} \right\}$$

with $e^{-i\omega t}$ time dependence where $\omega > 0$, the time-dependent Maxwell equations reduce to the time-harmonic system

$$\begin{aligned} -i\omega\, \hat{\mathbf{B}} + \mathbf{curl}\, \hat{\mathbf{E}} &= \mathbf{0} \\ \mathrm{div}\, \hat{\mathbf{D}} &= \hat{\rho} \\ i\omega\, \hat{\mathbf{D}} + \mathbf{curl}\, \hat{\mathbf{H}} &= \hat{\mathbf{J}} \\ \mathrm{div}\, \hat{\mathbf{B}} &= 0. \end{aligned}$$

Assumptions

- Ω_+ consists of homogeneous, isotropic material. In this case we have

$$\mathbf{D} = \varepsilon\,\mathbf{E} \qquad \mathbf{B} = \mu\,\mathbf{H},$$

  where ε and μ are constants and are called, respectively, the *electric permittivity* and *magnetic permeability*.

- Ω_- is a three-dimensional perfectly conducting object with regular bounded surface Γ.

- No external sources i.e. $\mathbf{J} = \mathbf{0}$ and $\rho = 0$.

Exterior scattering problem

Find $(\mathbf{E}, \mathbf{H})$ such that

$$-\varepsilon \frac{\partial \mathbf{E}}{\partial t} + \mathbf{curl}\,\mathbf{H} = \mathbf{0} \quad \text{in } \mathbb{R}_t \times \Omega_+$$

$$\mu \frac{\partial \mathbf{H}}{\partial t} + \mathbf{curl}\,\mathbf{E} = \mathbf{0} \quad \text{in } \mathbb{R}_t \times \Omega_+$$

$$\operatorname{div} \mathbf{E} = \operatorname{div} \mathbf{H} = 0 \quad \text{in } \mathbb{R}_t \times \Omega_+$$

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with boundary conditions

$$\mathbf{n} \times \mathbf{E} = -\mathbf{n} \times \mathbf{E}^{\mathrm{inc}} \quad \text{on } \mathbb{R}_t \times \Gamma$$

and initial conditions

$$\mathbf{E}(t, \mathbf{x}) = \mathbf{H}(t, \mathbf{x}) = \mathbf{0} \quad \text{for } t \leq 0 \text{ and } \mathbf{x} \in \Omega_+.$$

Integral representation (1)

For $\mathbf{y} \in \Omega_+$ the solution can be expressed in terms of a retarded boundary integral

$$\mathbf{E}(t,\mathbf{y}) = -\mu \int_0^t \int_\Gamma k(t-\tau,\, \mathbf{x}-\mathbf{y})\, \partial_\tau \mathbf{j}(\tau,\mathbf{x})\, d\Gamma_\mathbf{x}\, d\tau$$
$$- \frac{1}{\varepsilon} \nabla \int_0^t \int_\Gamma k(t-\tau,\, \mathbf{x}-\mathbf{y})\, q(\tau,\mathbf{x})\, d\Gamma_\mathbf{x}\, d\tau$$

where $\mathbf{j}$ is the unknown surface current density, q is the unknown surface charge density and the kernel k is given by

$$k(t,\mathbf{z}) := \frac{\delta(t - \sqrt{\varepsilon\mu}\|\mathbf{z}\|)}{4\pi\|\mathbf{z}\|}.$$

Integral representation (2)

The unknown boundary densities $\mathbf{j}$ and q are determined via the boundary condition. Let ∇_Γ denote the surface gradient and let $y \to \Gamma$. Then we obtain the equation

$$\mu \int_0^t \int_\Gamma k(t-\tau, \mathbf{x}-\mathbf{y})(\mathbf{n_y} \times \mathbf{n_y} \times \partial_\tau \mathbf{j}(\tau, \mathbf{x}))\, d\Gamma_{\mathbf{x}} d\tau \\ - \frac{1}{\varepsilon} \nabla_\Gamma \int_0^t \int_\Gamma k(t-\tau, \mathbf{x}-\mathbf{y}) q(\tau, \mathbf{x})\, d\Gamma_{\mathbf{x}} d\tau = -\mathbf{n_y} \times \mathbf{c}(t, \mathbf{y})$$

for $t \in \mathbb{R}$ and $\mathbf{y} \in \Gamma$ where $\mathbf{c} = -\mathbf{n} \times \mathbf{E}^{\mathrm{inc}}$.

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for $t \in \mathbb{R}$ and $\mathbf{y} \in \Gamma$ where $\mathbf{c} = -\mathbf{n} \times \mathbf{E}^{\text{inc}}$.

Remark: $\mathbf{j}$ and q are related by the law of conservation of charge

$$\frac{\partial q}{\partial t} + \text{div}_\Gamma \mathbf{j} = 0.$$

Convolution quadrature

For the time discretization of the above boundary integral we employ Lubich's convolution quadrature method. Consider

$$V(\partial_t)\phi(t) := \int_0^t v(t-\tau)\phi(\tau)\,d\tau, \qquad 0 \le t \le T$$

where V denotes the Laplace transform of v. Introducing a time step $\Delta t = T/N$, $N > 0$ and $t_n = n\Delta t$ we seek an approximation of the form

$$V(\partial_t^{\Delta t})\phi(t_n) := \sum_{j=0}^{n} \omega_{n-j}^{\Delta t}(V)\phi(t_j) \qquad \text{for } n = 0, \dots, N.$$

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Remark. In our application v is a parameter dependent integral operator.

Convolution quadrature

Convolution quadrature is based on the Laplace transform and makes use of the Laplace inversion formula

$$v(t) = \frac{1}{2\pi i} \int_{\sigma + i\mathbb{R}} V(s) e^{st} \, ds.$$

Inserting the above formula into the convolution integral leads to

$$V(\partial_t)\phi(t) = \frac{1}{2\pi i} \int_{\sigma + i\mathbb{R}} V(s) \int_0^t e^{s(t-\tau)} \phi(\tau) \, d\tau ds.$$

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The inner integral is the solution of the ordinary differential equation

$$y'(t) = sy(t) + \phi(t), \; y(0) = 0.$$

Convolution quadrature

We approximate the solution of this ODE by a linear multistep method of the form

$$\sum_{j=0}^{k} \alpha_j \, y_{n+j-k}(s) = \Delta t \sum_{j=0}^{k} \beta_j (s \, y_{n+j-k}(s) + \phi((n+j-k)\Delta t))$$

with $y_n(s) \approx y(s, t_n)$ and $y_{-k}(s) = \ldots = y_{-1}(s) = 0$. We also assume that ϕ equals zero on the negative real axis.

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It turns out that a sub-class of A-stable methods are desirable. Candidates include:

- Backward Euler
- BDF2

Convolution quadrature

Multiplying by ξ^n, summing over n from 0 to ∞ and rearranging terms leads to the formal power series

$$\sum_{n=0}^{\infty} y_n \xi^n = \left(\frac{\gamma(\xi)}{\Delta t} - s \right)^{-1} \sum_{n=0}^{\infty} \phi(n\Delta t)\xi^n$$

where $\gamma(\xi) := \frac{\sum_{j=0}^{k} \alpha_j \xi^{k-j}}{\sum_{j=0}^{k} \beta_j \xi^{k-j}}$ is the quotient of the generating polynomials of the underlying multistep method.

For the BDF2 scheme we have

$$\gamma(\xi) = \frac{1}{2}\xi^2 - 2\xi + \frac{3}{2}.$$

Convolution quadrature

Therefore we get

$$\sum_{n=0}^{\infty} V(\partial_t)\phi(t_n)\xi^n \approx \frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}} V(s)\left(\frac{\gamma(\xi)}{\Delta t} - s\right)^{-1} \sum_{n=0}^{\infty} \phi(n\Delta t)\xi^n \, ds$$

$$= V\left(\frac{\gamma(\xi)}{\Delta t}\right) \sum_{n=0}^{\infty} \phi(n\Delta t)\xi^n$$

by Cauchy's integral formula. Expanding $V\left(\frac{\gamma(\xi)}{\Delta t}\right)$ in a formal Taylor series at $\xi = 0$ defines coefficients $\omega_m^{\Delta t}(V)$ s.t.

$$V\left(\frac{\gamma(\xi)}{\Delta t}\right) = \sum_{m=0}^{\infty} \omega_m^{\Delta t}(V)\,\xi^m.$$

Convolution quadrature

This finally leads to a discrete convolution

$$V(\partial_t^{\Delta t})\phi(t_n) = \sum_{j=0}^{n} \omega_{n-j}^{\Delta t}(V)\,\phi(t_j)$$

where $n = 0, 1, \ldots, N$.

Convolution quadrature

This finally leads to a discrete convolution

$$V(\partial_t^{\Delta t})\phi(t_n) = \sum_{j=0}^{n} \omega_{n-j}^{\Delta t}(V)\,\phi(t_j)$$

where $n = 0, 1, \ldots, N$.

In our application we want to find the unknown density function ϕ s.t.

$$V(\partial_t)\phi(t) = g(t) \qquad \text{for } t \in [0, T]$$

Using convolution quadrature we approximate the solution by solving

$$\sum_{j=0}^{n} \omega_{n-j}^{\Delta t}(V)\,\phi^{\Delta t}(t_j) = g(t_n)$$

for $n = 0, 1, \ldots, N$.

Convolution quadrature

We follow the approach of Banjai and Sauter and transfer this Toeplitz system to the Fourier image.

In order to do that we represent the quadrature weights as a contour integral

$$\omega_m^{\Delta t}(V) = \frac{1}{2\pi i} \oint_C \frac{V(\gamma(\xi)/\Delta t)}{\xi^{m+1}}\, d\xi$$

where C is a circle centered at the origin of radius $\rho < 1$. Next, we apply the trapezoidal rule and get the approximate weights

$$\omega_m^{\Delta t,\rho}(V) = \frac{\rho^{-m}}{N+1} \sum_{l=0}^{N} V(s_l)\zeta_{N+1}^{lm}$$

with $\zeta_{N+1} = e^{\frac{2\pi i}{N+1}}$ and $s_l = \gamma\left(\rho\zeta_{N+1}^{-l}\right)/\Delta t$.

Convolution quadrature

Thus, we approximate the solution by solving

$$\sum_{j=0}^{n} \omega_{n-j}^{\Delta t,\rho}(V)\,\phi^{\Delta t,\rho}(t_j) = g(t_n)$$

for $n = 0, \ldots, N$. This is equivalent to

$$\frac{\rho^{-n}}{N+1} \sum_{l=0}^{N} \left(V(s_l)\hat{\phi}_l^{\Delta t,\rho} \right) \zeta_{N+1}^{ln} = g(t_n)$$

for $n = 0, \ldots, N$, where $\hat{\phi}_l^{\Delta t,\rho}$ is a scaled discrete Fourier transform

$$\hat{\phi}_l^{\Delta t,\rho} = \sum_{j=0}^{N} \rho^j \phi^{\Delta t,\rho}(t_j) \zeta_{N+1}^{-lj}.$$

Convolution quadrature

We apply the scaled discrete Fourier transform on both sides and get the following decoupled system of time-independent problems

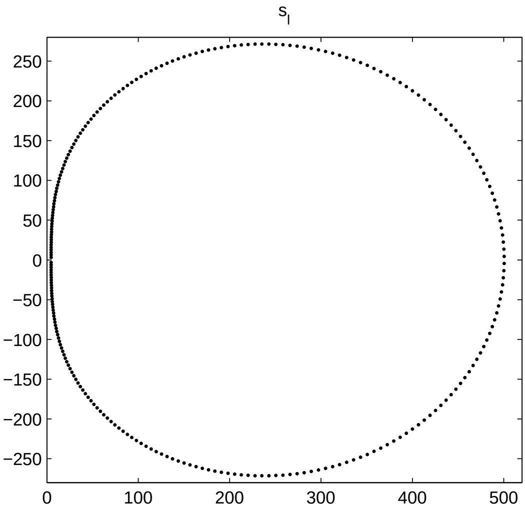
$$V(s_l)\hat{\phi}_l^{\Delta t,\rho} = \hat{g}_l$$

for $l = 0, \ldots, N$.

We obtain the time-domain solution by applying the scaled inverse transform

$$\phi^{\Delta t,\rho}(t_n) = \frac{\rho^{-n}}{N+1} \sum_{l=0}^{N} \hat{\phi}_l^{\Delta t,\rho} \zeta_{N+1}^{nl}.$$

Convolution quadrature



A range of complex frequencies s_l for $N = 256$, $T = 2$ and $\rho^N = 10^{-4}$. In this case we have $\mathrm{Re}(s_l) > 4.6$ for $l = 1, \ldots, N$.

Convolution quadrature for Maxwell's equations

We have to solve the boundary integral equation

$$\mu \int_0^t \int_\Gamma k(t-\tau, \mathbf{x}-\mathbf{y})(\mathbf{n_y} \times \mathbf{n_y} \times \partial_\tau \mathbf{j}(\tau, \mathbf{x}))\, d\Gamma_\mathbf{x} d\tau \\ - \frac{1}{\varepsilon} \nabla_\Gamma \int_0^t \int_\Gamma k(t-\tau, \mathbf{x}-\mathbf{y}) q(\tau, \mathbf{x})\, d\Gamma_\mathbf{x} d\tau = -\mathbf{n_y} \times \mathbf{c}(t, \mathbf{y})$$

for $t \in \mathbb{R}$ and $\mathbf{y} \in \Gamma$ where $\mathbf{c} = -\mathbf{n} \times \mathbf{E}^{\text{inc}}$.

In the next step we will use the relation

$$q(t, \mathbf{x}) = -\int_0^t \mathrm{div}_\Gamma \mathbf{j}(\tau, \mathbf{x})\, d\tau$$

and employ the Laplace transform to make this equation suitable for convolution quadrature.

Convolution quadrature for Maxwell's equations

The boundary integral equation is equivalent to

$$\mu \int_0^t \int_\Gamma \mathcal{L}^{-1}\left\{s\, K(s,\mathbf{x}-\mathbf{y})\right\}(t-\tau)\left[\mathbf{n}_\mathbf{y}\times\mathbf{n}_\mathbf{y}\times\mathbf{j}(\tau,\mathbf{x})\right] d\Gamma_\mathbf{x} d\tau$$

$$+\frac{1}{\varepsilon}\nabla_\Gamma \int_0^t \int_\Gamma \mathcal{L}^{-1}\left\{\frac{1}{s}\, K(s,\mathbf{x}-\mathbf{y})\right\}(t-\tau)\ \mathrm{div}_\Gamma \mathbf{j}(\tau,\mathbf{x})\, d\Gamma_\mathbf{x} d\tau = -\mathbf{n}_\mathbf{y}\times\mathbf{c}$$

where K is the Laplace transform of the time domain kernel function

$$K(s,\mathbf{z}) := \mathcal{L}\left\{k\right\}(s,\mathbf{z}) = \frac{e^{-s\sqrt{\varepsilon\mu}\|\mathbf{z}\|}}{4\pi\|\mathbf{z}\|}$$

Convolution quadrature for Maxwell's equations

This representation allows us to apply convolution quadrature and the derived formula. Thus we have solve the following system of time-harmonic problems at different complex wavenumbers s_l

$$\mu \int_\Gamma s_l\, K(s_l, \mathbf{x}-\mathbf{y})\, [\mathbf{n_y} \times \mathbf{n_y} \times \hat{\mathbf{j}}_l(\mathbf{x})]\, d\Gamma_\mathbf{x} + \frac{1}{\varepsilon} \nabla_\Gamma \int_\Gamma \frac{1}{s_l}\, K(s_l, \mathbf{x}-\mathbf{y})\, \mathrm{div}_\Gamma \hat{\mathbf{j}}_l(\mathbf{x})\, d\Gamma_\mathbf{x} = -\mathbf{n_y} \times \hat{\mathbf{c}}_l(\mathbf{y})$$

for all $\mathbf{y} \in \Gamma$ and $l = 0, \ldots, N$.

Involved Sobolev spaces

We define the following Hilbert space and its associated graph norm

$$\mathbf{H}(\mathbf{curl}, \Omega) := \{\mathbf{v} \in L^2(\Omega)^3, \mathbf{curl}\, \mathbf{v} \in L^2(\Omega)^3\}$$

$$\|\mathbf{v}\|_{\mathbf{curl}} = (\|\mathbf{v}\|_0^2 + \|\mathbf{curl}\, \mathbf{v}\|_0^2)^{1/2}$$

where $\|\cdot\|_0$ denotes the usual norm in $L^2(\Omega)^3$. Furthermore we introduce the following spaces on Γ:

$$\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \left\{\mathbf{v} \in \mathbf{H}^{-1/2}(\Gamma),\ \mathbf{n} \,.\, \mathbf{v} = 0,\ \mathrm{div}_\Gamma \mathbf{v} \in H^{-1/2}(\Gamma)\right\}$$

$$\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) := \left\{\mathbf{v} \in \mathbf{H}^{-1/2}(\Gamma),\ \mathbf{n} \,.\, \mathbf{v} = 0,\ \mathrm{curl}_\Gamma \mathbf{v} \in H^{-1/2}(\Gamma)\right\}$$

Trace theorem

Theorem

The trace mapping

$$\begin{aligned}\gamma_\times : \boldsymbol{H}(\boldsymbol{curl}, \Omega) &\rightarrow \boldsymbol{H}^{-1/2}(div_\Gamma, \Gamma) \\ \boldsymbol{v} &\mapsto [\boldsymbol{v} \times \boldsymbol{n}]_{|\Gamma}\end{aligned}$$

is continuous and surjective. Moreover, there exists a continuous lifting for the trace operator in $\boldsymbol{H}(\boldsymbol{curl}, \Omega)$.

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Lemma

If we identify $L^2(\Gamma)$ *with its dual space,* $\boldsymbol{H}^{-1/2}(div_\Gamma, \Gamma)$ *is the dual space of* $\boldsymbol{H}^{-1/2}(curl_\Gamma, \Gamma)$ *and conversely.*

Norms

The Lemma above shows that a norm of $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ is given by

$$\|\mathbf{u}\|_{-1/2,\mathrm{div}} = \sup_{\boldsymbol{\varphi}\in\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)} \frac{\left|\int_\Gamma \mathbf{u}\,.\,\boldsymbol{\varphi}\,d\Gamma\right|}{\|\boldsymbol{\varphi}\|_{-1/2,\mathrm{curl}}}.$$

A norm of $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ is given by

$$\|\mathbf{v}\|_{-1/2,\mathrm{curl}} = \sup_{\boldsymbol{\varphi}\in\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \frac{\left|\int_\Gamma \mathbf{v}\,.\,\boldsymbol{\varphi}\,d\Gamma\right|}{\|\boldsymbol{\varphi}\|_{-1/2,\mathrm{div}}}.$$

Variational formulation

Define $R(s) : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ by

$$R(s)\mathbf{j} := \mu \int_\Gamma s\, K(s_l, \mathbf{x} - \mathbf{y})\, [\mathbf{n_y} \times \mathbf{n_y} \times \mathbf{j}(\mathbf{x})]\, d\Gamma_\mathbf{x}$$

$$+ \frac{1}{\varepsilon} \nabla_\Gamma \int_\Gamma \frac{1}{s} K(s, \mathbf{x} - \mathbf{y})\ \mathrm{div}_\Gamma \mathbf{j}(\mathbf{x})\, d\Gamma_\mathbf{x}$$

An appropriate variational formulation for the system of time-harmonic Maxwell equations is given by:

Find $\hat{\mathbf{j}}_l \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ s.t. for all $\mathbf{q} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ holds

$$\int_\Gamma \left(-R(s_l)\hat{\mathbf{j}}_l(\mathbf{y}), \mathbf{q}(\mathbf{y}) \right) d\Gamma = \int_\Gamma \left(\hat{\mathbf{E}}_l^{\mathrm{inc}}(\mathbf{y}), \mathbf{q}(\mathbf{y}) \right) d\Gamma$$

for $l = 0, \ldots, N$.

Basic estimates

Theorem

For $s \in \mathbb{C}$ with $Re(s) \geq \sigma_0 > 0$ the sesquilinear form is continuous and coercive on $\boldsymbol{H}^{-1/2}(div_\Gamma, \Gamma) \times \boldsymbol{H}^{-1/2}(div_\Gamma, \Gamma)$. In particular

$$Re\left(\int_\Gamma \boldsymbol{\varphi} \,.\, (-\overline{R(s)\boldsymbol{\varphi}})\, d\Gamma\right) \geq C\, \frac{1}{|s|^2}\, \|\boldsymbol{\varphi}\|^2_{-1/2,\, div}$$

for all $\boldsymbol{\varphi} \in \boldsymbol{H}^{-1/2}(div_\Gamma, \Gamma)$.

For the inverse operator $R(s)^{-1}$ we have

$$\|R(s)^{-1}\| \leq \tilde{C}|s|^2$$

where $\tilde{C}$ depends on σ_0.

Current and Future work

- Implementation of the method
- Error analysis
- Numerical tests