computational mathematics

Convergence Analysis for Finite Element Discretization of the Helmholtz Equation

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Formulation of the Model Problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. We consider Helmholtz equation

$$-\Delta u - k^2 u = f \qquad in \quad \Omega,$$

with real and positive wave number

$$k \geq k_0 > 0,$$

and Robin boundary condition

$$\frac{\partial u}{\partial \mathbf{n}} + iku = g \qquad on \quad \partial\Omega.$$

Weak Form

So the weak form is given by

$$\text{Find} \quad u \in H^1(\Omega) \qquad s.t. \qquad a(u,v) = F(v) \qquad \forall v \in H^1(\Omega),$$

where

$$a(u,v) := \int_\Omega < \nabla u, \nabla \bar{v} > -k^2 u\bar{v} + ik \int_{\partial\Omega} u\bar{v},$$

$$F(v) := \int_\Omega f\bar{v} + \int_{\partial\Omega} g\bar{v}.$$

Galerkin Discretization

Let $S \subset H^1(\Omega)$ be a finite dimensional space.

$$\text{Find} \quad u_S \in S \qquad s.t. \qquad a(u_S, v) = F(v) \qquad \forall v \in S.$$

Existence, uniqueness and convergence are proved (Melenk, Diss, 1995) under the stability condition

$$\dim S \gtrsim k^{2d}. \tag{1}$$

Question: Is there subspace S such that condition (1) can be replaced by much weaker condition

$$\dim S \gtrsim k^{d}.$$

Adjoint Approximation Property

$$\eta(S) := \sup_{f \in L^2(\Omega)} \inf_{v \in S} \frac{\| N_k^* f - v \|_{\mathcal{H}}}{\| f \|_{L^2(\Omega)}},$$

where

$$\| u \|_{\mathcal{H}} := \left(|u|^2_{H_1(\Omega)} + k^2 \| u \|^2_{L^2(\Omega)} \right)^{1/2}.$$

and N_k^* is the solution operator of the adjoint problem.

Theorem

Let $\Omega = B_R$ be a ball with radius $R \quad (R \geq R_0 > 0)$. Assume that the space S is chosen such that

$$k\eta(S) \leq \frac{1}{4C_c}.$$

Then, the discrete inf-sup condition satisfies

$$\inf_{u \in S} \sup_{v \in S \setminus \{0\}} \frac{|a(u,v)|}{\| u \|_{\mathcal{H}} \| v \|_{\mathcal{H}}} \geq \frac{1}{2 + C_c^{-1} + 4k}$$

and this ensures existence and uniqueness of the discrete problem.

Convergence Analysis

Theorem

Let the assumptions of previous theorem be satisfied. Let u denote the solution of weak form and u_s its Galerkin approximation. Then

$$\| u - u_S \|_{\mathcal{H}} \leq 2C_c \inf_{v \in S} \| u - v \|_{\mathcal{H}} .$$

The L^2-error satisfies

$$\| u - u_S \|_{L^2(B_R)} \leq C_c \eta(S) \| u - u_S \|_{\mathcal{H}} .$$

Analysis of Adjoint Approximation Property

Model Problem:
Find $u \in H^1(\Omega)$ such that

$$(-\Delta - k^2)u = f \qquad in \quad \Omega,$$

$$\frac{\partial u}{\partial \mathbf{n}} = Tu \qquad on \quad \partial\Omega.$$

The Dirichlet-to-Neumann map is given by

$$T := \gamma_1 S_p : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega),$$

where $\gamma_1 := \frac{\partial}{\partial \mathbf{n}}$ is the normal trace operator.

Consider the exact solution of the model problem

$$U(x) := N_k f(x) := \int_\Omega G_k(x-y) f(y) dy \qquad \forall x \in \Omega,$$

where

$$G_k : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$$

$$G_k(z) = g_k(\| z \|).$$

Here $g_k(r) = \frac{e^{ikr}}{2\pi r}$ is fundamental solution to the Helmholtz equation for $d = 3$.

Decomposition Lemma

Lemma

Let $\Omega \subset B_R$*, for* $f \in L^2(\Omega)$ *the solution of our model problem is*

$$v(x) := N_k f(x) := \int_\Omega G_k(x-y) f(y) dy \qquad \forall x \in \Omega.$$

It satisfies

$$\| v \|_{\mathcal{H}} \leq C \| f \|_{L^2(\Omega)} .$$

For any $\lambda > 1$*, there exists an* λ*-dependent splitting* $v = v_{H^2} + v_{\mathcal{A}}$ *with*

$$\| \nabla^p v_{H^2} \|_{L^2(\Omega)} \leq C(\lambda k)^{p-2} \| f \|_{L^2(\Omega)} \qquad \forall p = 0,1,2,$$

$$\| \nabla^p v_{\mathcal{A}} \|_{L^2(\Omega)} \leq C(\gamma \lambda k)^{p-1} \| f \|_{L^2(\Omega)} \qquad \forall p \in \mathbb{N}_0.$$

Corollary

In case $p = 2$ we have:

$$\| v_{H^2} \|_{H^2(\Omega)} \leq C \| f \|_{L^2(\Omega)},$$
$$\| v_{\mathcal{A}} \|_{H^2(\Omega)} \leq Ck \| f \|_{L^2(\Omega)} .$$

Definition

For given $\lambda > 1$, $\gamma > 0$, and $k \geq k_0 > 0$ let

$$\mathcal{H}^{osc}(\gamma, k) := \{v \in H^1(\Omega) : \| \nabla^p v \|_{L^2(\Omega)} \leq (\gamma \lambda k)^{p-1} \quad \forall p \in \mathbb{N}_0\},$$

$$\mathcal{H}^{H^2} := \{v \in H^2(\Omega) : \quad \| v \|_{H^2(\Omega)} \leq 1\}.$$

The approximation properties for the oscillatory and the H^2-part are

$$\eta_{\mathcal{A}}(S, k) := \sup_{v \in \mathcal{H}^{osc}(\gamma, k)} \inf_{w \in S} \| v - w \|_{\mathcal{H}}$$

$$\eta_{H^2}(S) := \sup_{v \in \mathcal{H}^{H^2}} \inf_{w \in S} \| v - w \|_{\mathcal{H}} .$$

Corollary

Let $d \in \{1, 2, 3\}$ and assume the hypotheses of the Decomposition lemma. Let $S \subset H^1(\Omega)$ be a finite dimensional approximation space. Then

$$\eta(S) \leq C_{k,\mathcal{A}} \eta_{\mathcal{A}}(S) + C_{H^2} \eta_{H^2}(S).$$

Hence, the estimate of the adjoint approximation property $\eta(S)$ is boiled down to estimate of the approximation properties of a finite element space S for

(a) H^2-functions and

(b) highly oscillatory, analytic functions.

hp-FEM

Let $\Omega \subset \mathbb{R}^d$ is a bounded domain with analytic boundary. $\mathcal{T} = \{\tau_i : \quad 1 \leq i \leq q\}$ is a conforming finite element triangulation, and we define

$$h_\tau := diam\, \tau, \qquad h_{\mathcal{T}} := \max\{h_\tau : \tau \in \mathcal{T}\}.$$

Hp-finite element space:

$$S^p_{\mathcal{T}} := \{u \in H^1(\Omega) \quad | \quad \forall \tau \in \mathcal{T} : u|_\tau \circ F_\tau \in \mathbb{P}_p\},$$

where $\mathbb{P}_p$ is the space of polynomials of degree p.

Theorem

Let $\Omega = B_R$, there exist constants c_1 and c_2 independent of k, h, and p such that

$$\frac{kh}{p} \leq c_1 \qquad \textit{together with} \qquad p \geq c_2 \ln k,$$

implies the existence and uniqueness of the finite element solution.

The minimal dimension of the corresponding $h - \log k$ finite element space satisfies

$$\dim S = O(k^d).$$

Theorem

Let the assumption of the previous theorem be satisfied. Then, the discrete problem has unique solution. It holds

$$\| u - u_S \|_{\mathcal{H}} \leq C_c\left(\frac{h}{p} + \left(\frac{kh}{\sigma p}\right)^p\right) \| f \|_{L^2(\Omega)} .$$

Approximation Properties

The estimates of $\eta_{H^2}(S)$, $\eta_{\mathcal{A}}(S)$ and $\inf_{w \in S} \parallel u - w \parallel_{\mathcal{H}}$ are approximation properties for $H^2(\Omega)$ functions and highly oscillatory functions in $\mathcal{H}^{osc}(\gamma, k)$.

For hp-finite element spaces these estimates are derived by using suitable polynomial interpolation operators.

Thank you for your attention.