Robust Maxwell Formulation

Florian Krämer, Seminar for Applied Mathematics, ETH Zürich

Overview

- Introduction
- Robust Full Maxwell Formulation
  - Problems with standard full Maxwell formulation
  - Stabilization
  - Some Examples
- Operator Precondition
  - Introduction
  - Application to Robust Maxwell Formulation
  - Numerical Examples
- $\mathcal{H}$-Matrix technique
  - Introduction
  - Application to Robust Maxwell Formulation & Eddy Current Model
  - Numerical Examples

Introduction

- supported by ABB Research Center in Baden-Dättwil
- CAD integrated software
- used in daily business
- modularized program, HADAPT

Maxwell Model Hierarchy

$$\begin{aligned} \mathbf{curl}\,\mathbf{e} &= -i\omega\mu\mathbf{h} \\ \mathbf{curl}\,\mathbf{h} &= \sigma\mathbf{e} + i\omega\epsilon\mathbf{e} \end{aligned} \qquad \omega \gg 1$$

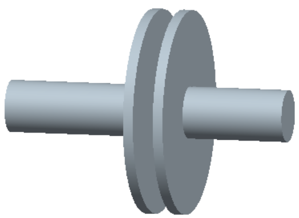
Full Maxwell, wave phenomena

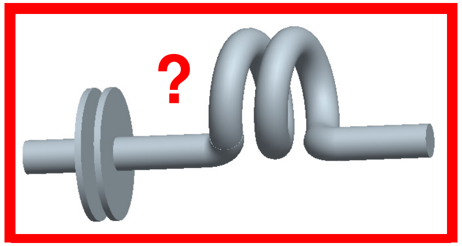
$$\begin{aligned} \mathbf{curl}\,\mathbf{e} &= -\cancel{i\omega\mu}\mathbf{h} \\ \mathbf{curl}\,\mathbf{h} &= \sigma\mathbf{e} + i\omega\epsilon\mathbf{e} \end{aligned} \qquad \omega \ll \frac{c}{L} \qquad \begin{aligned} \mathbf{curl}\,\mathbf{e} &= -i\omega\mu\mathbf{h} \\ \mathbf{curl}\,\mathbf{h} &= \sigma\mathbf{e} + \cancel{i\omega\epsilon}\mathbf{e} \end{aligned}$$

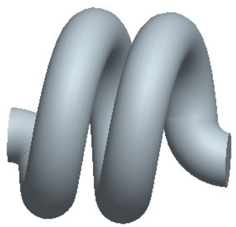
Electro-quasistatics | Magneto-quasistatics

$$\mathbf{curl}\,\mathbf{e} = -\cancel{i\omega\mu}\mathbf{h} \qquad \omega = 0 \qquad \mathbf{curl}\,\mathbf{h} = \sigma\mathbf{e} + \cancel{i\omega\epsilon}\mathbf{e}$$

Electro-statics | Magneto-statics







Most important quantities

electric field $\mathbf{e} = -\mathbf{grad}\,\varphi - i\omega\mathbf{a}$

current density $\mathbf{j} = \sigma \cdot \mathbf{e}$

electric displacement field $\mathbf{d} = \epsilon\mathbf{e}$ $\qquad \epsilon \approx 10^{-12}$

magnetic field $\mathbf{b} = \mathbf{curl}\,\mathbf{a}$

magnetizing field $\mathbf{h} = \frac{1}{\mu}\mathbf{b}$ $\qquad \mu \approx 10^{-7}$

$\mathbf{a}$ vector potential for magnetic field

φ scalar potential for electric field

μ $\mu_r\mu_0$ = relative permeability · absolute permeability

ϵ $\epsilon_r\epsilon_0$ = relative Permittivity · absolute Permittivity

ω angular frequency $= 2\pi \cdot$ frequency

Full Maxwell System in Frequency Domain

$$\begin{aligned} \textbf{curl}\,\frac{1}{\mu}\textbf{curl}\,\mathbf{a} - (\omega^2\epsilon - i\omega\sigma)\mathbf{a} + (i\omega\epsilon + \sigma)\,\mathbf{grad}\,\varphi &= 0 \\ -\,\mathrm{div}\,(\epsilon\mathbf{a}) &= 0 \end{aligned}$$

$\mathbf{a}$ vector potential for magnetic field

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μ $\mu_r\mu_0 =$ relative permeability · absolute permeability

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Full Maxwell System in Frequency Domain

$$
\begin{aligned}
\mathbf{curl}\,\frac{1}{\mu}\mathbf{curl}\,\mathbf{a} - \cancel{(\omega^2\epsilon} \cancel{- i\omega\sigma)}\mathbf{a} + (\cancel{i\omega\epsilon +} \sigma)\,\mathbf{grad}\,\varphi &= 0 \\
-\operatorname{div}(\epsilon\mathbf{a}) &= 0
\end{aligned}
$$

$\omega \to 0$ to get the stationary case or to calculate low-frequency

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$\omega \to 0$ to get the stationary case or to calculate low-frequency

$\sigma = 0$ For non-conducting materials e.g. in the airbox

Full Maxwell System in Frequency Domain

$$\begin{aligned} \mathbf{curl}\,\frac{1}{\mu}\mathbf{curl}\,\mathbf{a} - (\omega^2\epsilon - i\omega\sigma)\mathbf{a} + (i\omega\epsilon + \sigma)\mathbf{grad}\,\varphi &= 0 \\ -\operatorname{div}(\epsilon\mathbf{a}) &= 0 \end{aligned}$$

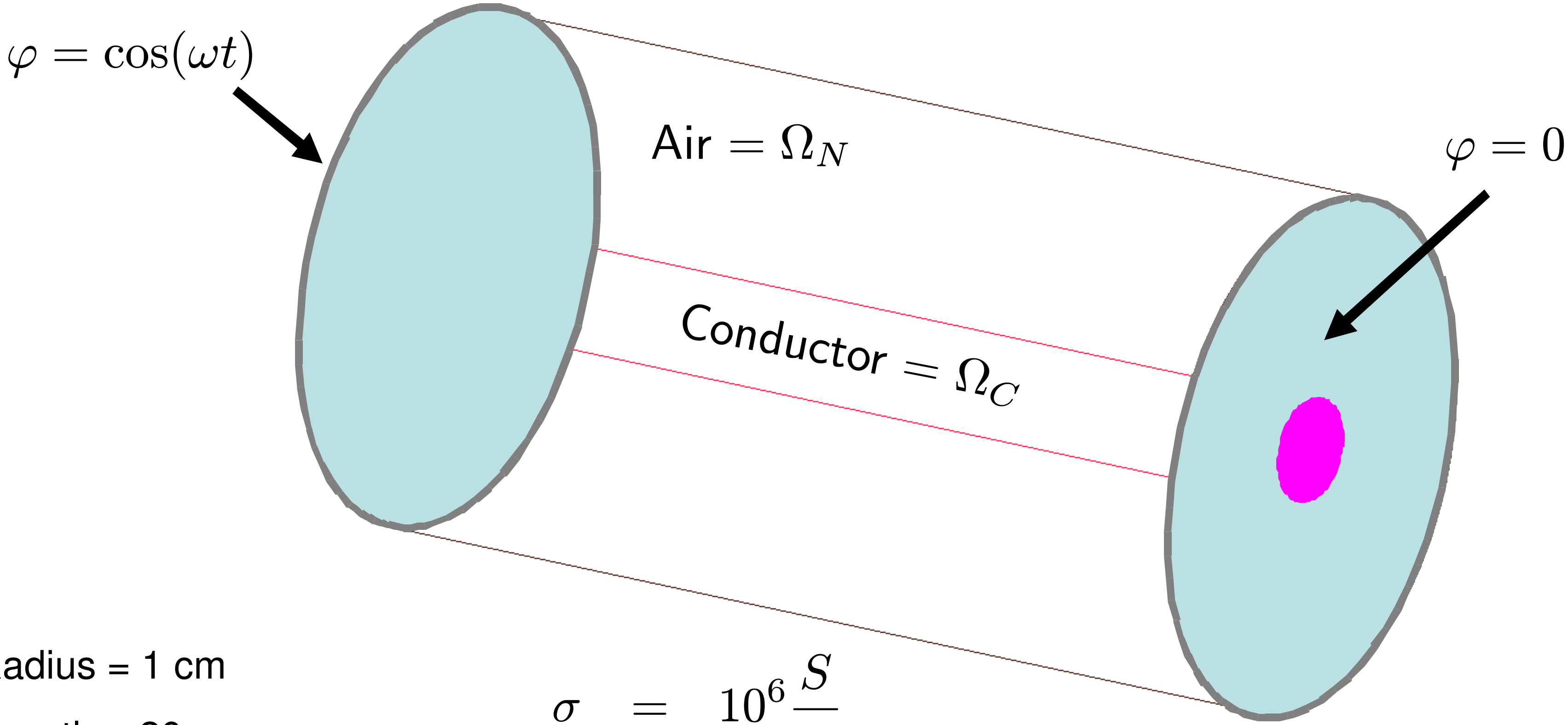
(On the slide, the terms $\omega^2\epsilon$, $i\omega\sigma$ and $i\omega\epsilon$ are crossed out in blue, and σ in $(i\omega\epsilon + \sigma)$ is crossed out in red.)

$\omega \to 0$ to get the stationary case or to calculate low-frequency

$\sigma = 0$ For non-conducting materials e.g. in the airbox

Loss of control of the electric potential φ!

Numerical experiments: Geometry for Frequency test



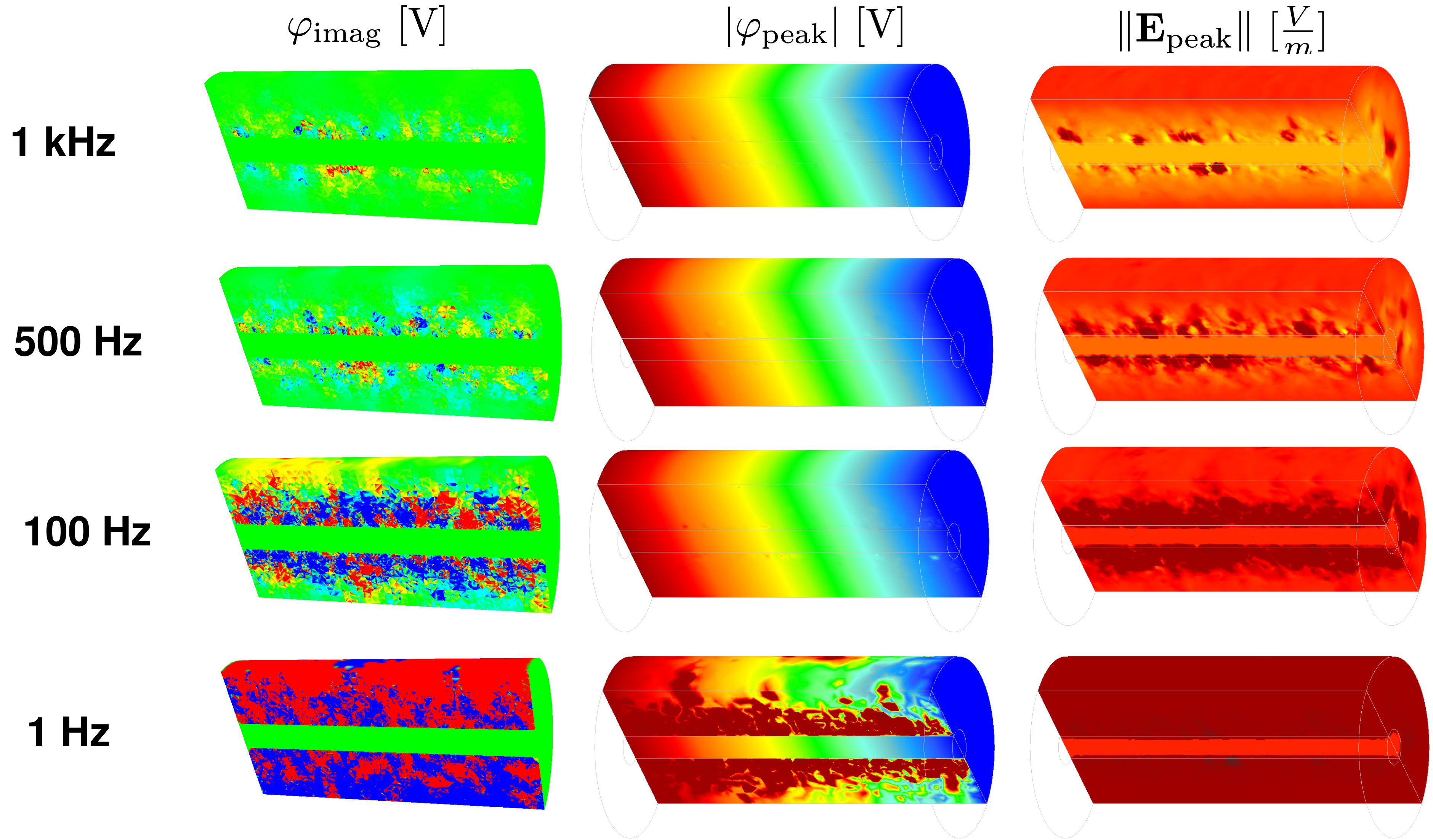
Radius = 1 cm

Length = 20 cm

$$\sigma = 10^6 \frac{S}{m}$$

$$\mu_r = \epsilon_r = 1$$

Numerical experiments: Frequency test



Stabilization: Extendend $\mathbf{a}$-φ-formulation

$$\begin{aligned} \mathbf{curl}\,\frac{1}{\mu}\mathbf{curl}\,\mathbf{a} - (\omega^2\epsilon - i\omega\sigma)\mathbf{a} + (i\omega\epsilon + \sigma)\mathbf{grad}\,\varphi &= 0 \\ \operatorname{div}(\epsilon\mathbf{a}) &= 0 \end{aligned}$$

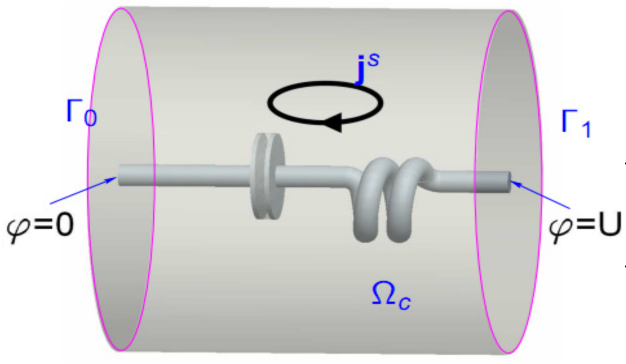
Apply div in non-conducting domain Ω_N

$\rightarrow$ time derivative of Gauss law ($\operatorname{div}\,\mathbf{e} = \frac{\rho}{\epsilon}$)

$$i\omega\,\operatorname{div}(i\omega\epsilon\mathbf{a} + \epsilon\mathbf{grad}\,\varphi) = 0 \Leftrightarrow \dot{\rho} = 0$$

Idea: integrate Gauss law by splitting $\varphi = \tilde{\varphi} + \psi$ in Ω_N

Robust Full Maxwell Formulation



$$V := \{v \in H(\textbf{curl}\,;\Omega) : \textbf{curl}_{\Gamma}\mathbf{v}_t = 0 \text{ on } \partial\Omega\},$$

$$H(U) := \{\psi \in H^1(\Omega) : \psi_{|\Gamma_0} = 0, \psi_{|\Gamma_1} = U\},$$

$$H^1_{\mathsf{e}}(\Omega) := \left\{ \begin{array}{l} v \in H^1(\Omega) : v \text{ ´ } \text{ const on con.} \\ \text{comp. of } \Omega_C, v_{|\Gamma_0} = 0, v_{|\Gamma_1} = 0 \end{array} \right\}$$

Stabilized variational formulation:

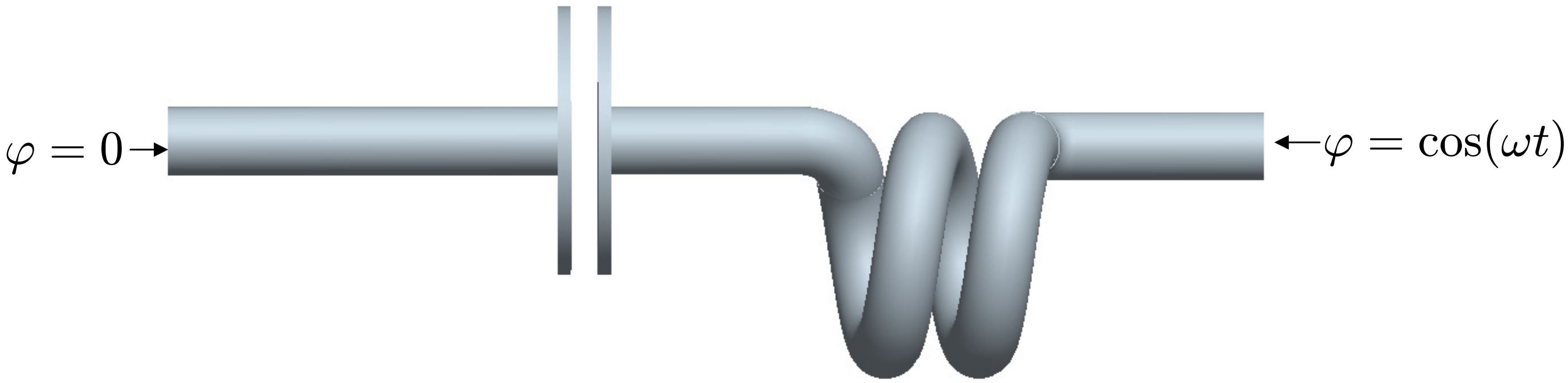
$$\left\langle \frac{1}{\mu}\textbf{curl}\,\mathbf{a}, \textbf{curl}\,\mathbf{a}' \right\rangle - \langle (\omega^2\epsilon - i\omega\epsilon)\mathbf{a}, \mathbf{a}' \rangle$$

$$+ \langle (i\omega\epsilon + \sigma)\,\mathbf{grad}\,\varphi, \mathbf{a}' \rangle + \langle i\omega\epsilon\mathbf{grad}\,\psi, a' \rangle = \langle \mathbf{j}^s, a \rangle \qquad \forall \mathbf{a}' \in V \qquad (1)$$

$$\langle \epsilon\mathbf{a}, \mathbf{grad}\,\tilde{\varphi}' \rangle = 0 \qquad \forall \tilde{\varphi}' \in H(0) \qquad (2)$$

$$\langle \epsilon\mathbf{grad}\,[\tilde{\varphi} + \psi]\,, \mathbf{grad}\,\psi' \rangle = 0 \qquad \forall \psi' \in H^1_e(\Omega) \qquad (3)$$

Setting: LC-circuit



diameter 2.54 cm

length 40 cm

capacitor diamter 10 cm

capacitor distance 1 cm

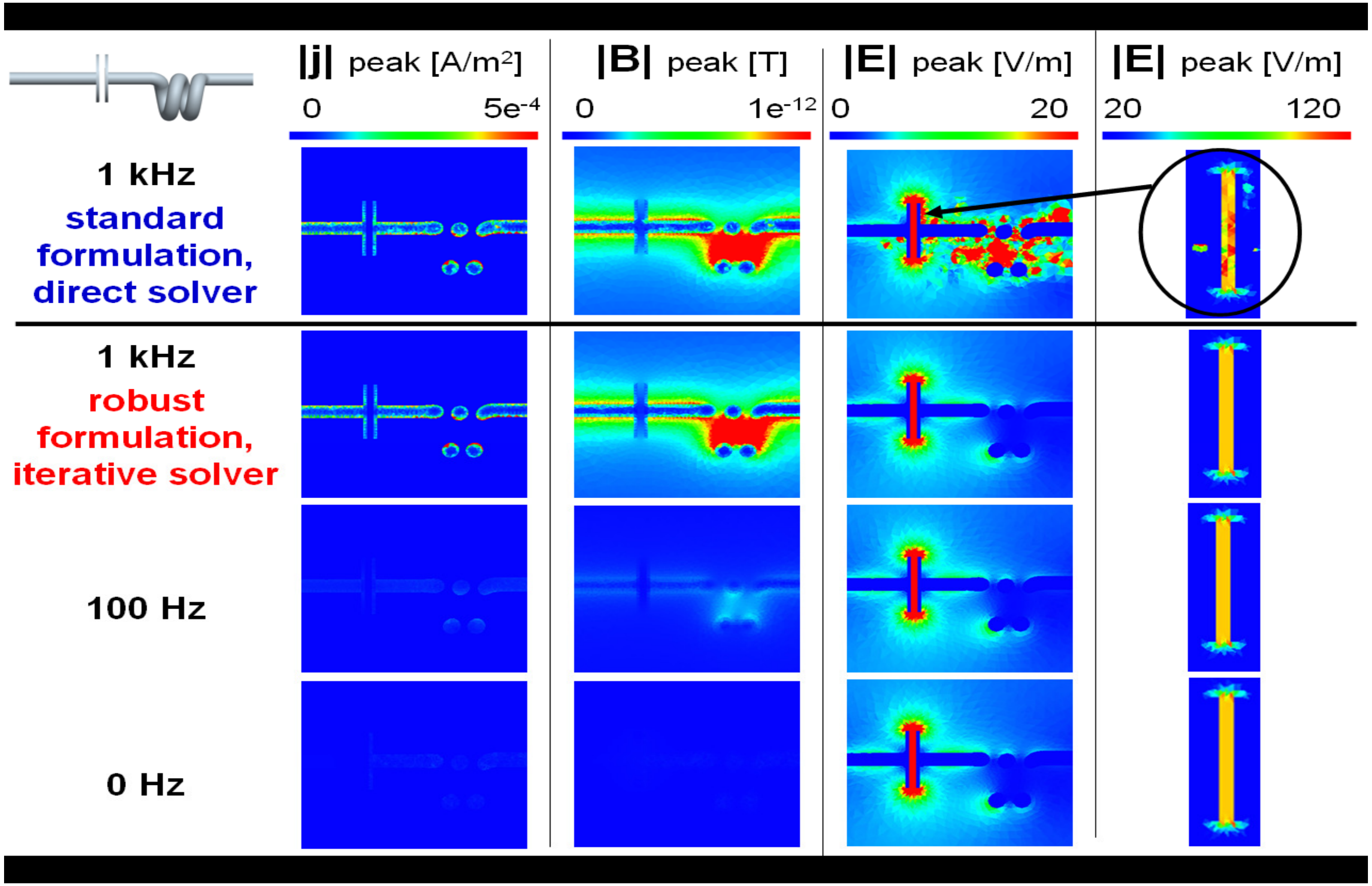
Material: Copper

σ $5.7 \cdot 10^7 \frac{S}{m}$

μ_r 1

ϵ_r 1

Robust Maxwell Formulation: Numerical Examples



Operator Preconditioning

Stabilized variational formulation:

$$\left\langle \frac{1}{\mu}\textbf{curl}\,\mathbf{a}, \textbf{curl}\,\mathbf{a}' \right\rangle - \left\langle \left(\omega^2\epsilon - i\omega\epsilon\right)\mathbf{a}, \mathbf{a}' \right\rangle$$

$$+ \langle (i\omega\epsilon + \sigma)\,\mathbf{grad}\,\varphi, \mathbf{a}' \rangle + \langle i\omega\epsilon\mathbf{grad}\,\psi, a' \rangle \quad = \quad \langle j^s, a \rangle \qquad \forall \mathbf{a}' \in V \qquad (1)$$

$$\langle \epsilon a, \mathbf{grad}\,\tilde{\varphi}' \rangle = 0 \qquad \forall \tilde{\varphi}' \in H(0) \qquad (2)$$

$$\langle \epsilon\mathbf{grad}\,[\tilde{\varphi} + \psi]\,, \mathbf{grad}\,\psi\prime \rangle = 0 \qquad \forall \psi' \in H^1_e(\Omega) \qquad (3)$$

Wish: Other system with better properties e.g. real and s.p.d.

Problem: The new system has still to limit/reduce the number of iteration steps e.g.

Requirement: Choose a system that captures the essential properties of the stiffness matrix of the robust full Maxwell system.

Operator Preconditioning

Stabilized variational formulation:

$$\begin{aligned}
&\left\langle \frac{1}{\mu}\mathbf{curl}\,\mathbf{a}, \mathbf{curl}\,\mathbf{a}' \right\rangle - \left\langle \left(\omega^2\epsilon - i\omega\epsilon\right)\mathbf{a}, \mathbf{a}' \right\rangle \\
&+ \langle (i\omega\epsilon + \sigma)\,\mathbf{grad}\,\varphi, \mathbf{a}' \rangle + \langle i\omega\epsilon\mathbf{grad}\,\psi, a' \rangle \quad = \quad \langle j^s, a \rangle \qquad \forall \mathbf{a}' \in V \\
&\langle \epsilon a, \mathbf{grad}\,\tilde{\varphi}' \rangle \quad = \quad 0 \qquad \forall \tilde{\varphi}' \in H(0) \\
&\langle \epsilon\mathbf{grad}\,[\tilde{\varphi} + \psi]\,, \mathbf{grad}\,\psi' \rangle \quad = \quad 0 \qquad \forall \psi' \in H^1_e(\Omega)
\end{aligned}$$

$$\begin{pmatrix}
\mathbf{curl}\,\frac{1}{\mu}\mathbf{curl}\, + \sigma\omega & 0 & 0 \\
0 & -\operatorname{div}(\epsilon\mathbf{grad}\,) & 0 \\
0 & 0 & -\operatorname{div}(\epsilon\mathbf{grad}\,)_{|\Omega_C}
\end{pmatrix}$$

Operator Preconditioning

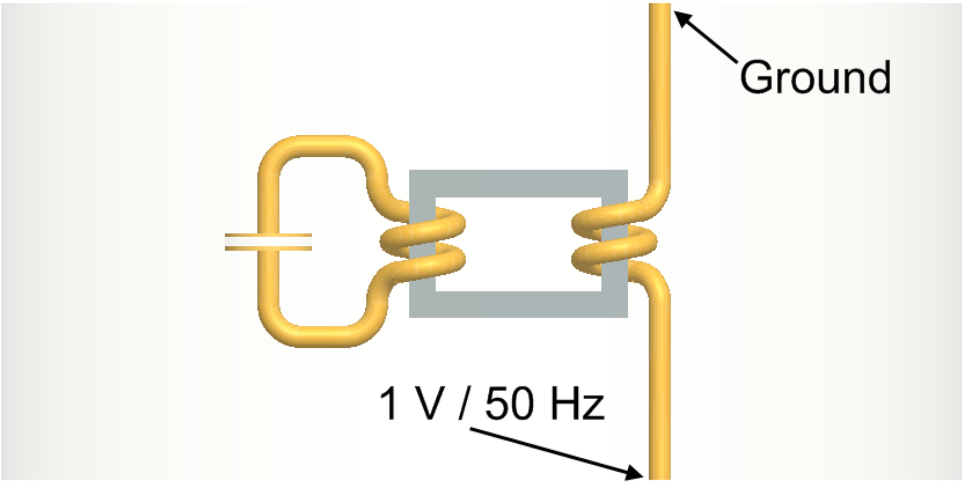
Stabilized variational formulation:

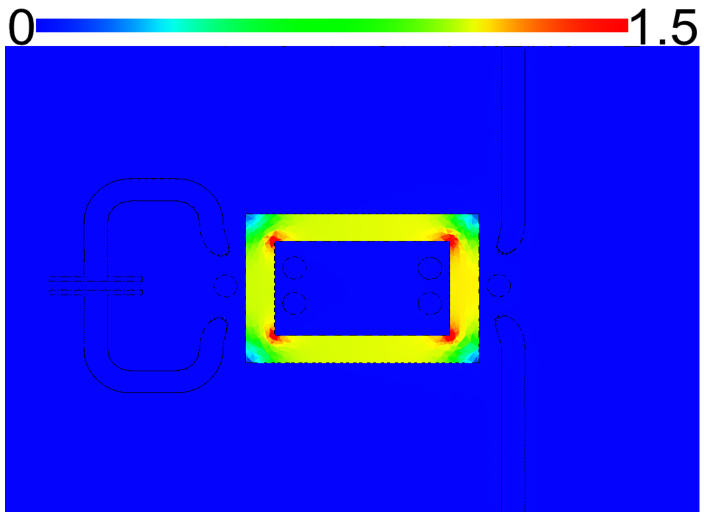
$$
\begin{aligned}
\left\langle \frac{1}{\mu}\mathbf{curl}\,\mathbf{a}, \mathbf{curl}\,\mathbf{a}' \right\rangle - \langle (\omega^2\epsilon - i\omega\epsilon)\mathbf{a}, \mathbf{a}' \rangle & & & \\
+ \langle (i\omega\epsilon + \sigma)\,\mathbf{grad}\,\varphi, \mathbf{a}' \rangle + \langle i\omega\epsilon\mathbf{grad}\,\psi, a' \rangle &= \langle j^s, a \rangle & & \forall \mathbf{a}' \in V \\
\langle \epsilon a, \mathbf{grad}\,\tilde{\varphi}' \rangle &= 0 & & \forall \tilde{\varphi}' \in H(0) \\
\langle \epsilon\mathbf{grad}\,[\tilde{\varphi} + \psi]\,, \mathbf{grad}\,\psi' \rangle &= 0 & & \forall \psi' \in H^1_e(\Omega)
\end{aligned}
$$

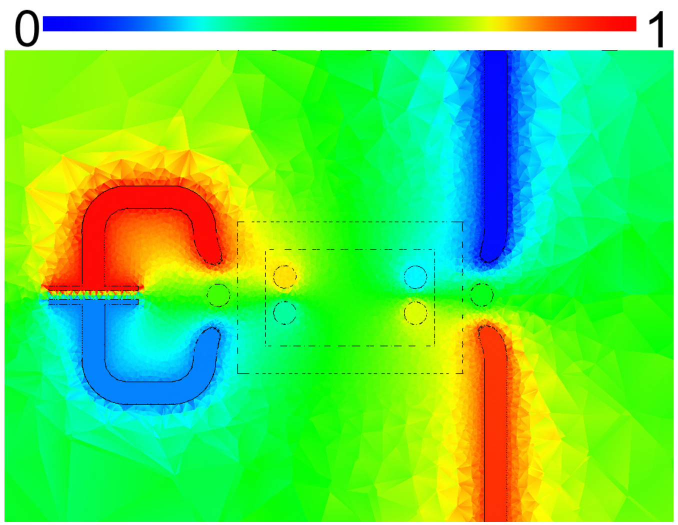
$$
\begin{pmatrix}
\mathbf{curl}\,\frac{1}{\mu}\mathbf{curl}\, + \sigma\omega & 0 & 0 \\
0 & -\operatorname{div}(\epsilon\mathbf{grad}\,) & 0 \\
0 & 0 & -\operatorname{div}(\epsilon\mathbf{grad}\,)_{|\Omega_C}
\end{pmatrix}
$$

Amenable to $\mathcal{H}$-Matrix preconditioning and is real and s.p.d.

Numerical Examples: Geometry







- high permeable core e.g. μ_r=1000
- inductive and capacative effects
- low frequency

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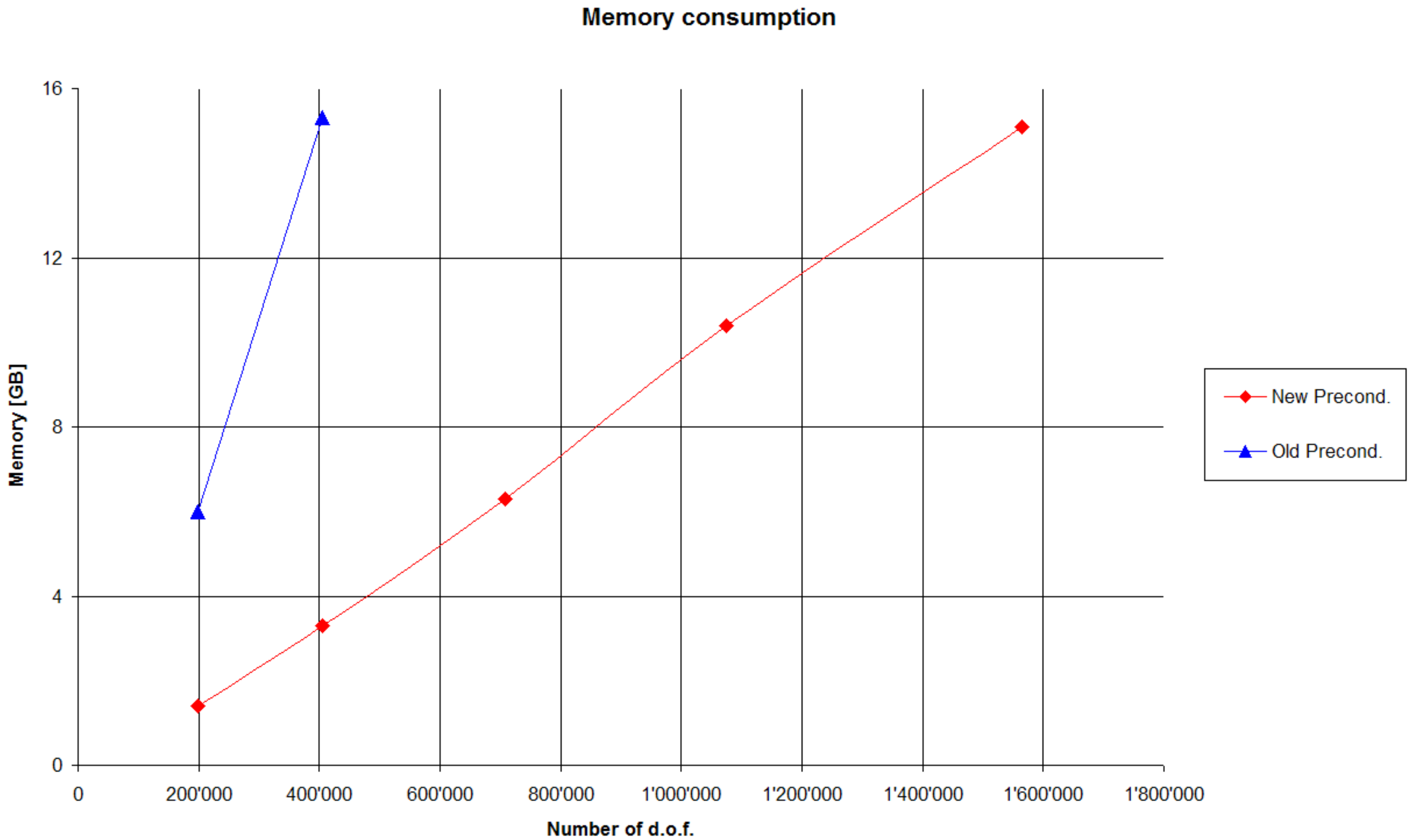
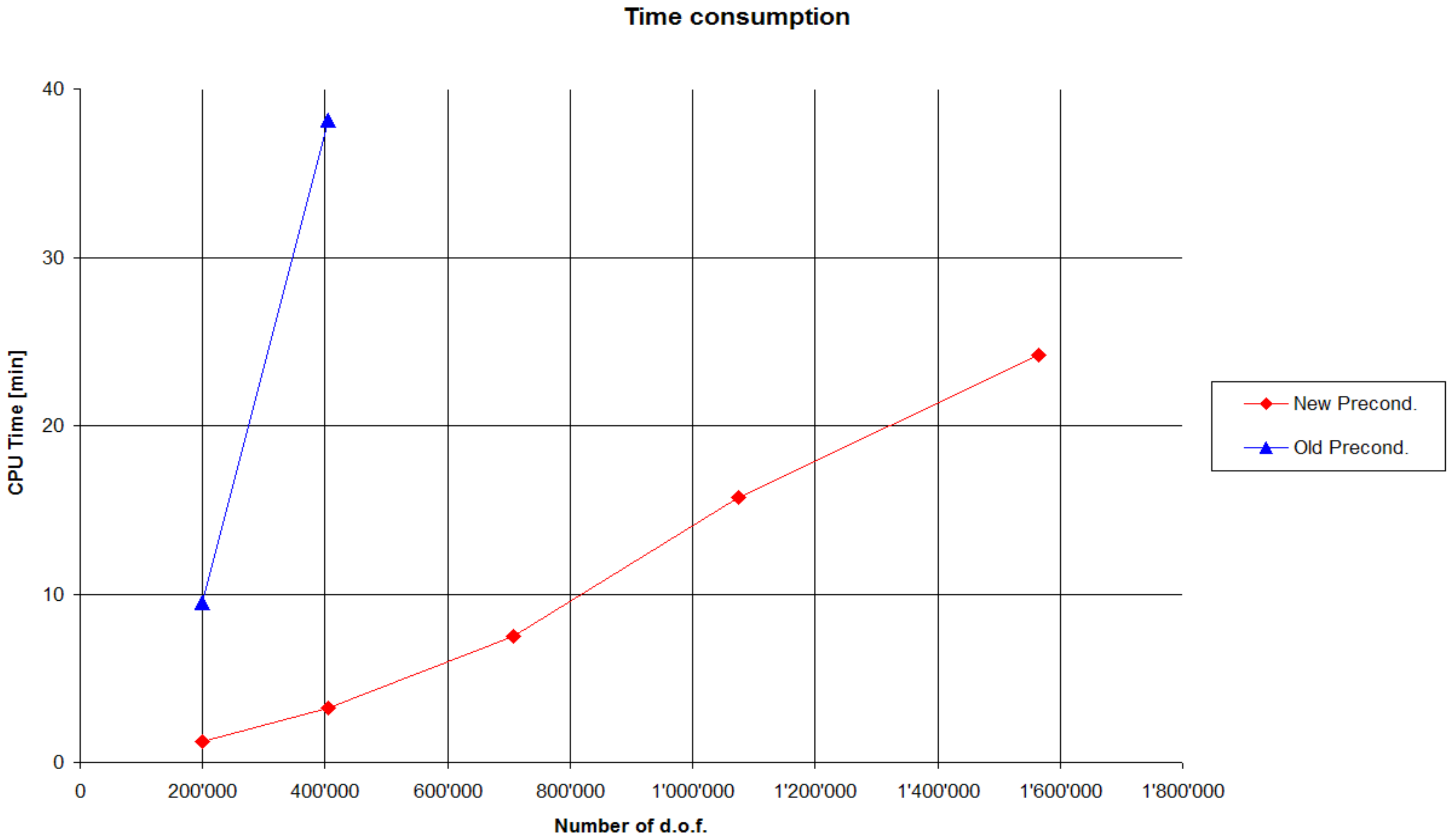
Operator Preconditioning: Memory consumption



Operator Preconditioning: Time consumption



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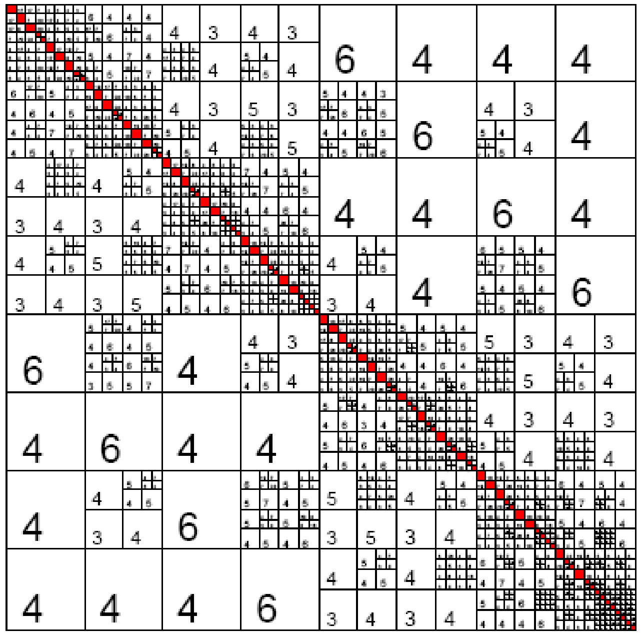
$\mathcal{H}$-Matrix:Introduction

- Uses two fundamental ideas:
  - Partion of the Matrix
  - Limitation to blockwise low-rank matrices



Partition:

$$P = \{b = t \times s, t, s \subset I\}$$

$$I := \{1, \ldots, n\}$$

with pair wise disjoint blocks b and

$$I \times I = \cup_{b \in P} b.$$

Matrix indices t are geometry related:

$$X_t := \cup_{i \in t} \text{supp}\, \varphi_i$$

$$t \times s \in P \Leftrightarrow \min\{\text{diam} X_t, \text{diam} X_s\} \leq \acute{}\, \text{dist}(X_t, X_s) \text{ or } t \times s \text{ is small.}$$

$\mathcal{H}$-Matrix:Introduction

$$H(P,k) := \left\{ M \in \mathbb{R}^{n\times n} : \text{rank}\, M_{|b} \leq k \quad \forall b \in P \right\}$$

Low-rank-matrices/rank-k Matrix $A \in \mathbb{R}^{m\times n}$:

- can be written as $A = UV', U \in \mathbb{R}^{m\times k}, V \in \mathbb{R}^{n\times k}$
- memory consumpotion of $k(m+n)$ instead of $m \cdot n$
- mv-multiplication is of Order $\mathcal{O}(k(m+n))$ instead of $\mathcal{O}(m \cdot n)$

$\mathcal{H}$-Matrix:Introduction

Complexity for $\mathcal{H}(P,k)$:

$$\begin{aligned} \text{Memory, mv-multi. :} &\quad kn\log n \\ A+B: &\quad k^2 n \log n \\ A\cdot B, A^{-1}, \text{LU}: &\quad k^2 n(logn)^2 \end{aligned}$$

where n denotes the number of unknowns.

- $\mathcal{H}$-mv-multiplication can be done without approximation.
- $\mathcal{H}$-matrix addition: blockweise truncated addition with precision ϵ.
- SVD of AB' of rank-k matrices with a QR decomposition of A and B (cost: $k^2(m+n)$) and the an SVD of $R_A R_B'$

$\mathcal{H}$-Matrix:FE-Application

✓ FE-Matrix is sparse

✘ bad condition of matrix

✘ Invese and LU is not sparse

- Use $\mathcal{H}$-Matrices to build a preconditioner
- $\mathcal{H}$-Matrices are robust for uniform-elliptic operators with jumping coe±cients
- It is possible to compute $\mathcal{H}$-Invese and $\mathcal{H}$-LU decomposition

$\mathcal{H}$-Matrices:Application

The problem with the application of $\mathcal{H}$-Matrices on the preconditioner is in the **curl** $\frac{1}{\mu}$**curl** Operator e.g. that the **curl** Operator has an infinite-dimensional kernel and that the problem is not elliptic.

In [1] was the $\mathcal{H}$-Matrix applied for the magneto-static problem

$$\textbf{curl}\,\frac{1}{\mu}\textbf{curl}\,\mathbf{a} = j_0 \text{ in } \Omega, \mathbf{a}\times n = 0 \text{ on } \partial\Omega$$

There was also shown, that it is necessary to regulize the magneto-static problem. Since the $\mathcal{H}$-Matrix was just used to compute a preconditioner, it was enough to apply it to

$$\textbf{curl}\,\frac{1}{\mu}\textbf{curl}\,\mathbf{a} + \beta\mathbf{a} = j_0 \text{ in } \Omega, \mathbf{a}\times n = 0 \text{ on } \partial\Omega$$

With $\beta \gg 1$

[1] M. Bebendorf and J. Ostrowski. Parallel hierarchical matrix preconditioners for the curl-curl operator. *Journal of Computational Mathematics, special issue on Adaptive and Multilevel Methods for Electromagnetics*, 27(5):624-641, 2009.

$\mathcal{H}$-Matrices:Application

$$\begin{pmatrix} \mathbf{curl}\,\frac{1}{\mu}\mathbf{curl} + \sigma\omega & 0 & 0 \\ 0 & -\operatorname{div}(\epsilon\,\mathbf{grad}) & 0 \\ 0 & 0 & -\operatorname{div}(\epsilon\,\mathbf{grad})_{|\Omega_C} \end{pmatrix}$$

Similiar to Laplace-Operator. No problem.

- Magneto-quasistatic operator
- was not regular enough, even with limiter

$\mathcal{H}$-Matrix:MQS Application

Problem: Has not worked for our **curl curl**-Operator:

$$\textbf{curl}\,\frac{1}{\mu}\textbf{curl}\,\mathbf{a} + \omega\sigma\mathbf{a} \tag{1}$$

Idea: Use a σ-adaptive regularization

When we denote the Matrix of (1) with P, then the new matrix is given by

$$P' = P + \alpha I$$

Where

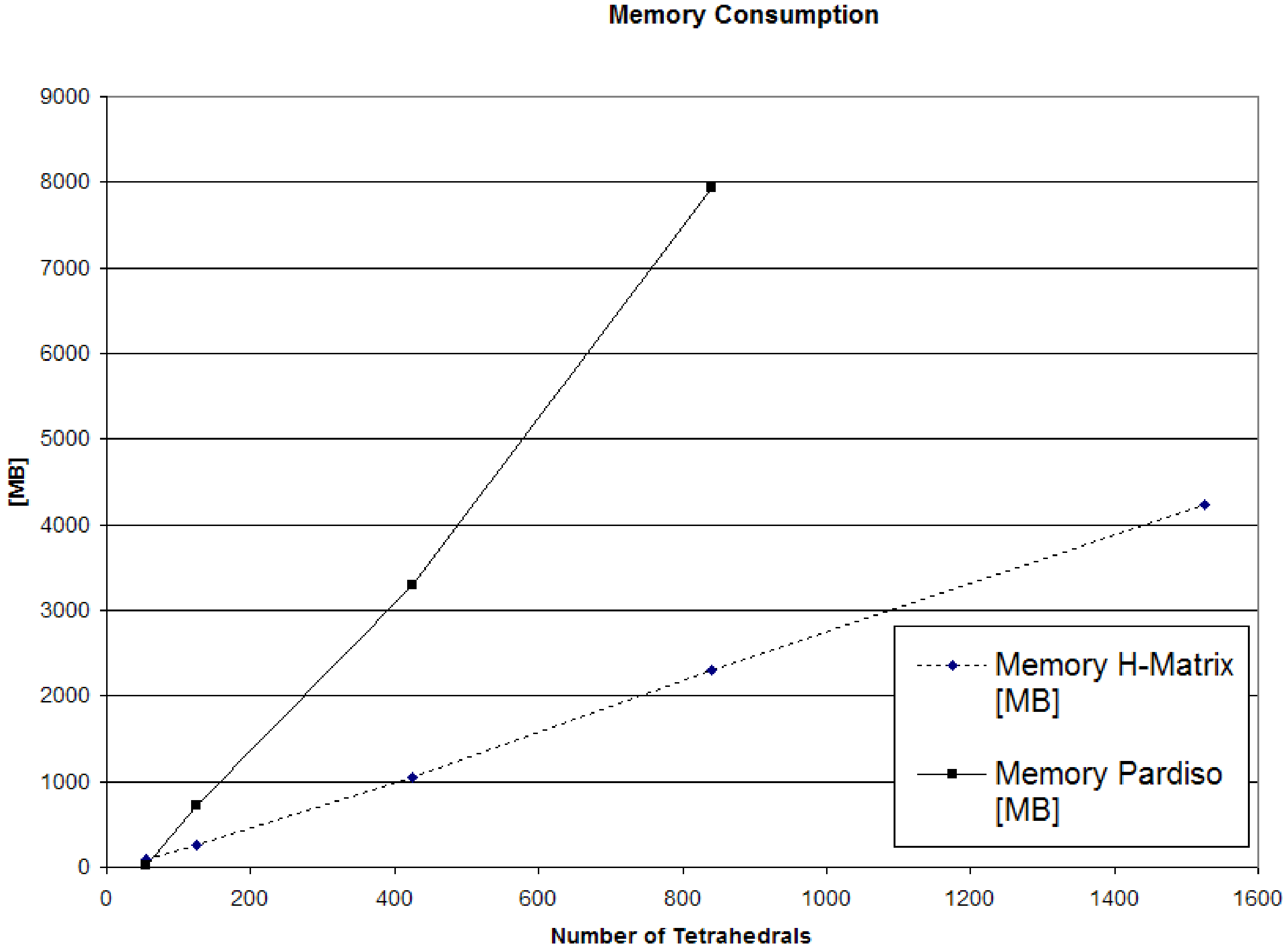
$$\alpha = \frac{\sum_{T\in C}\sigma(T)}{\text{card}(C)}$$

where C is the set of conductive tetrahedra.

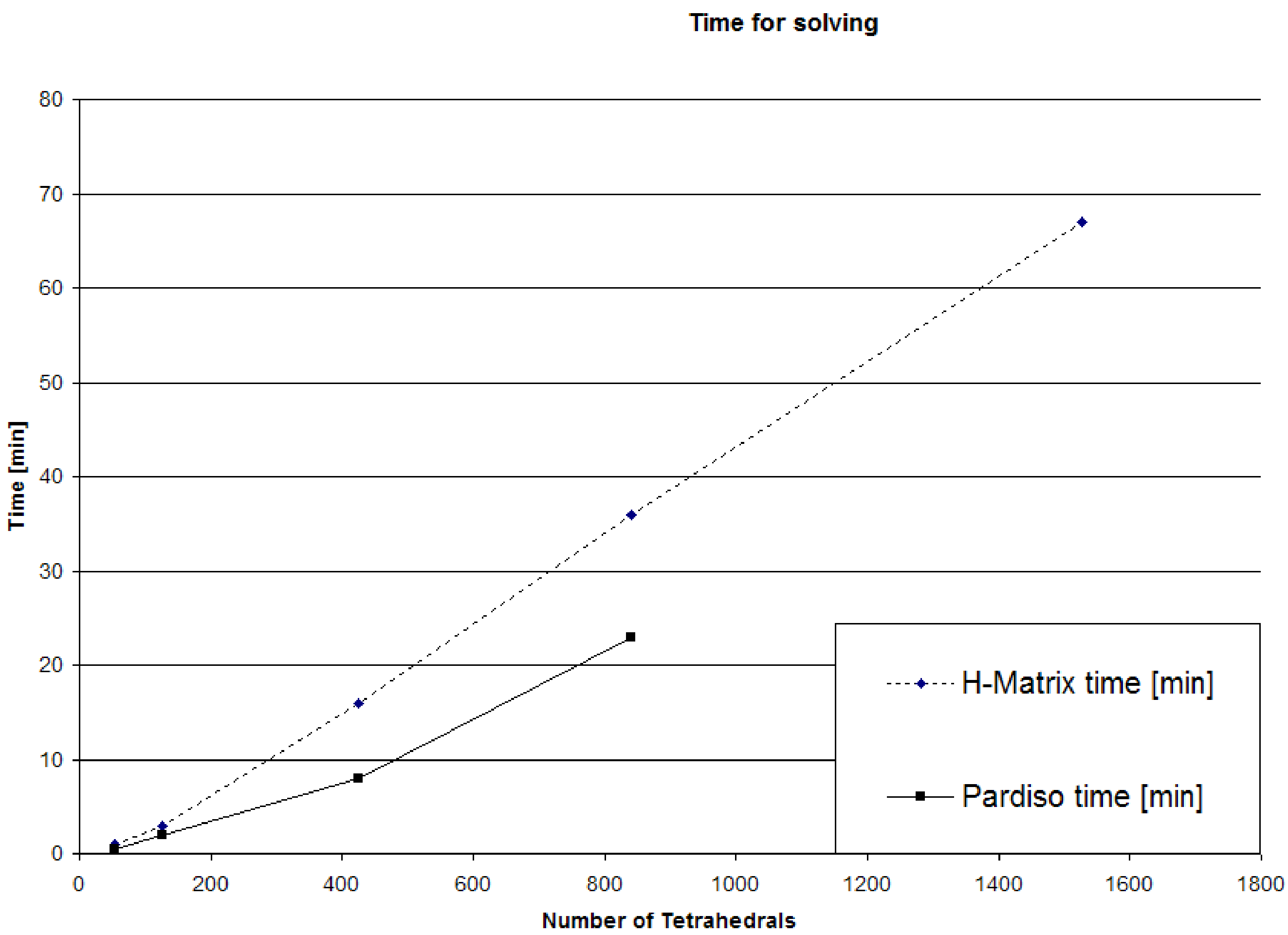
$\mathcal{H}$-Matrix:Numerical Experiments

- used σ-adaptive regularization
- computed $\mathcal{H}$-Choleksy decomposition with AHMED library and compared with direct solver PARDISO.
- we have just investigated the preconditiong effect on the magneto-quasistatic problem
- we have investigated the in˚uence on the performance when h or ω or μ_r is changed.
- PARDISO was always fasters, but has always allocated more memory and was due this not able to compute the largest examples

$\mathcal{H}$-Matrix:Numerical Experiments



$\mathcal{H}$-Matrix:Numerical Experiments



Conclusions

- Found new, robust formulation for Maxwell's equations for low frequencies. Now it is possible to compute inductive and capacitive effects if $\omega \to 0$ in an non-conductive domain.
- Improved preconditioner with operator preconditioning technique. Due to this trick we get a symmetric, positive definite matrix.
- Able to apply $\mathcal{H}$-matrices after we have found a way to compute the $\mathcal{H}$-matix approximation towards of the **curl curl** $+\sigma\omega$ part of the matrix

Thanks for your Attention!!!