ETH
Eidgenössische Technische Hochschule Zürich
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Seminar for Applied Mathematics SAM

Approximations of Kinetic Equations

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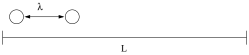
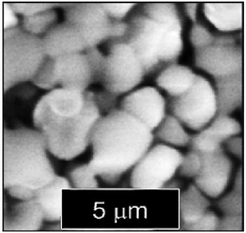
Modelling Beyond Equilibrium ...

- Particle **collisions** drive system into **equilibrium**
  $\longrightarrow$ equilibrium thermodynamics.
- Small scales $\longrightarrow$ **few** collisions
  $\longrightarrow$ extended, **non-equilibrium** thermodynamics

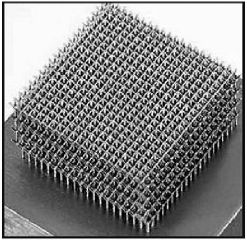
Scaling Parameter:

Knudsen Number $\varepsilon = \dfrac{\text{mean free path} \equiv \lambda}{\text{system size} \equiv L}$

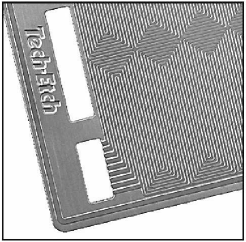




porous media



micro heat exchanger



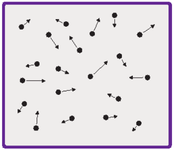
micro fuel cell

Kinetic Theory

- provides a general modeling framework for multi-scale **micro-macro** transitions:

Distribution function $f_{\mathbf{x},t}(\mathbf{c})$

Micro-World



mathematical transition

Macro-World

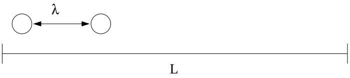
Fluid dynamic description with **fields** $\rho(\mathbf{x}.t)$, $\mathbf{v}(\mathbf{x},t)$, $\theta(\mathbf{x},t)$, etc.

- Microscopic:

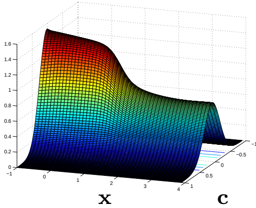
$$\underbrace{\partial_t f_{\mathbf{x},t}(\mathbf{c}) + c_k \partial_{x_k} f_{\mathbf{x},t}(\mathbf{c})}_{\text{advection}} = \overbrace{\frac{1}{\varepsilon}}^{\text{scale}} \underbrace{J(f,f)}_{\text{collisions}} \overset{\text{linearized}}{=} -\frac{1}{\varepsilon} K f$$

BOLTZMANN (1872)

- ⋆ $f_{\mathbf{x},t}(\mathbf{c})dc =$ gives **number density** of particles in $[\mathbf{c}, \mathbf{c} + d\mathbf{c}]$.
- ⋆ **Knudsen Number** $\varepsilon = \frac{\lambda}{L}$



1+1 dimensional example: $f_{\mathbf{x},t_0}(\mathbf{c})$



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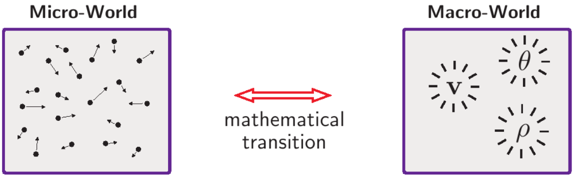

Fluid dynamic description with **fields** $\rho(\mathbf{x}.t)$, $\mathbf{v}(\mathbf{x},t)$, $\theta(\mathbf{x},t)$, etc.

- Macroscopic: (moment equations)

$$\rho(\mathbf{x},t) = m\int_{\mathbb{R}^3} f d\mathbf{c}, \qquad F_{i_1\ldots i_n}(t,\mathbf{x}) = m\int_{\mathbb{R}^3} c_{i_1}\ldots c_{i_n} f_{\mathbf{x},t}(\mathbf{c})d\mathbf{c}$$

Complexity reduction $f_{\mathbf{x},t}(\mathbf{c})$, $\mathbf{c}\in\mathbb{R}^3 \longleftrightarrow \{F_{i_1\ldots i_n}(\mathbf{x},t)\}_{n=0,1,\ldots,N}$.
Corresponds to special spectral method with monomials $c_{i_1}\ldots c_{i_n}$ as test functions.

$$\left.\begin{array}{rcl} \partial_t F + \partial_i F_i &=& 0\\ \partial_t F_i + \partial_j F_{ij} &=& 0\\ \partial_t F_{kk} + \partial_i F_{ikk} &=& 0 \end{array}\right\} \textbf{cons. laws}$$

$$\begin{array}{rcl} \partial_t F_{<ij>} + \partial_k F_{<ij>k} &=& P_{ij}\\ \partial_t F_{ijk} + \partial_l F_{ijkl} &=& P_{ijk}\\ \vdots & & \ddots \end{array}$$

Fluid Variables:
$F=\rho$, $F_i=\rho v_i$, $F_{kk}=3\rho\theta+\rho v_k^2$,
$F_{<ij>}\sim\sigma_{ij}$, $F_{ikk}\sim q_i$

Hierarchical Structure:
Flux(N) $\equiv$ variable$(N+1)$:
$\longrightarrow$ **Closure Problem.**

Kinetic Theory

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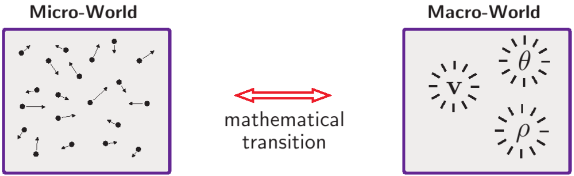


Fluid dynamic description with **fields** $\rho(\mathbf{x}.t)$, $\mathbf{v}(\mathbf{x},t)$, $\theta(\mathbf{x},t)$, etc.

- Transition: Modelling

$$F_{i_1 \dots i_N k} \approx \int_{\mathbb{R}^3} c_{i_1} \dots c_{i_N} c_k \mathbf{f}^{(\text{model})} \left(\lambda_\alpha[\mathbf{F}, \mathbf{F_i}, \dots, \mathbf{F_{i_1 \dots i_N}}](\mathbf{x}, \mathbf{t}), \mathbf{c}\right) d\mathbf{c}, \quad \alpha = 1, \dots, N,$$

λ_α any functional acting on $F_{i_1 \dots i_n}(\cdot, \cdot)$.

Goal: Find $f^{(\text{model})}$ such that resulting system

$$\partial_t U + \operatorname{div} \mathcal{F}(U) = \mathcal{P}(U)$$

- has nice mathematical properties (stability),
- and is physically accurate.

Low dimensional approximation of high dimensional problem.

Linearized Boltzmann

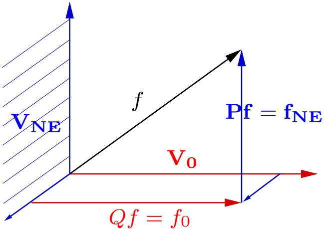
$$\partial_t f + \mathbf{c}\cdot\nabla f + \frac{1}{\varepsilon} K f = 0, \qquad f_{\mathbf{x},t}(\cdot) \in V, \qquad \text{Equilibrium: } \{f : Kf = 0\} = \ker(K) := V_0 \subset V.$$

Parametrization of Equilibrium

moments	$M : \mathbb{R}^m \ni \rho$	$\overset{\text{surjective}}{\longmapsto}$	$f_0 \in \ker K \equiv V_0$	**distribution**
distribution	$E_0 : V \ni f$	$\overset{\text{surjective}}{\longmapsto}$	$\rho \in \mathbb{R}^m$	**moments**

Features:

- $E_0 M = id_{\mathbb{R}^m \to \mathbb{R}^m}$.
- $V_{NE} := V \setminus V_0$: non-equilibrium phase space.
- $Q := ME_0 = id_{V_0} + 0_{V_NE} : V \to V_0$ (surjective) "euqilibrium projection".
- $P := id - Q = id_{V_NE} + 0_{V_0} : V \to V_{NE}$ (surjective) "non-equilibirum projection".



$$\partial_t f + c\partial_x f = -\frac{1}{\varepsilon} K f$$

Classical Closures

- Chapman-Enskog (1916) expansion in Knudsen number ε:

Ansatz :
$f = M\rho + \varepsilon f_1 + \varepsilon^2 f_2 + \mathcal{O}(\varepsilon^3)$.

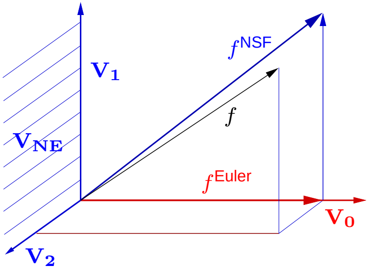
Euler / Navier-Stokes-Fourier equations

$$\partial_t \rho + E_0 \mathbf{c} \cdot \nabla M\rho = 0 + \varepsilon E_0 (\mathbf{c} \cdot \nabla) K^\dagger (\mathbf{c} \cdot \nabla) M\rho$$

$\xrightarrow{0^{\text{th}}\text{ order}}$ $f_1 = 0$ (Euler Equations)

$\xrightarrow{1^{\text{st}}\text{ order}}$ $f_1 = -K^\dagger \mathbf{c} \cdot \nabla M\rho$ (NSF)

$\xrightarrow{2^{\text{nd}}\text{ order}}$ Burnett Equations



⊖ 2^{nd} order unstable

⊕ accurate

Classical Closures

$$\partial_t f + c\partial_x f = -\frac{1}{\varepsilon} K f$$

♦ Chapman-Enskog (1916) expansion in Knudsen number ε:

Ansatz :
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Euler / Navier-Stokes-Fourier equations

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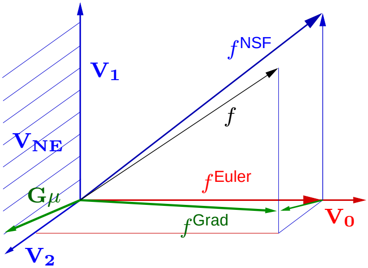
⊖ 2^{nd} order unstable ⊕ accurate



♦ Grad: assume $f = M\rho + G\mu$. $\mu \in W \subset \mathbb{R}^s$, $G : \mathbb{R}^s \to V_{NE}$ **arbitrary direction**.

Grad's equations (1949)

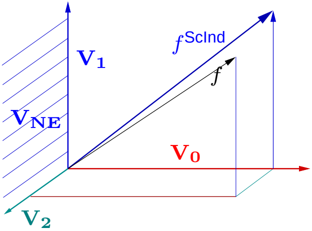
$$\begin{aligned} \partial_t \rho + E_0 \mathbf{c} \cdot \nabla M\rho + E_0 \mathbf{c} \cdot \nabla G\mu &= 0 \\ \partial_t \mu + E_1 \mathbf{c} \cdot \nabla M\rho + E_1 \mathbf{c} \cdot \nabla G\mu + \frac{1}{\varepsilon} E_1 K\, G\, \mu &= 0 \end{aligned}$$

⊕ stable ⊖ accuracy unclear

Scale Induced Closure [Torrilhon, Kauf, Levermore, Junk 2008; Struchtrup 2004]

Idea:
Combine **stability** of Grad and **accuracy** of Chapman-Enskog: $f = M\rho + \varepsilon G\mu + \varepsilon^2 f_2$.

Separation of phase space into $V = V_0 \oplus V_{NE} = \underbrace{V_0}_{\text{equilibrium}} \oplus \underbrace{V_1^{\varepsilon}}_{1^{\text{st}} \text{ order NE}} \oplus \underbrace{V_2^{\varepsilon^2}}_{2^{\text{nd}} \text{ order NE}} \oplus \underbrace{\dots}_{\text{higher NE}}$.



From kinetic equation: $G = -K^{\dagger}\mathbf{c}MD^{\dagger}$,
$D^{\dagger} : \mu \mapsto \nabla\rho$

Equations

$$\partial_t\rho + E_0\mathbf{c}\cdot\nabla M\rho - E_0\mathbf{c}\cdot\nabla G\mu = 0$$

$$\partial_t\mu + E_1\mathbf{c}\cdot\nabla M\rho - E_1\mathbf{c}\cdot\nabla G\mu - \frac{1}{\varepsilon}E_1 K\, G\mu = 0$$

$\oplus$ stable

$\oplus$ accurate to 2^{nd} order.

Scale Induced Closure

From kinetic equation: $G = -K^\dagger \mathbf{c} M D^\dagger$, $\quad D^\dagger : \mu \mapsto \nabla\rho$

Equations

$$\begin{aligned} \partial_t\rho + E_0\mathbf{c}\cdot\nabla M\rho - E_0\mathbf{c}\cdot\nabla G\mu &= 0 \\ \partial_t\mu + E_1\mathbf{c}\cdot\nabla M\rho - E_1\mathbf{c}\cdot\nabla G\mu - \frac{1}{\varepsilon}E_1 K\, G\mu &= 0 \end{aligned} \quad (1)$$

- Theorem 1: [Torrilhon, Kauf, Levermore, Junk 2008/2009]
  Under moderate conditions, the equations (1) are of 2^{nd} order if expanded in ε.
  $\longrightarrow$ **Generalization of Chapman-Enskog**

- Theorem 2: [Torrilhon, Kauf, Levermore, Junk 2008/2009]
  If K symmetric and positive semi definite, (1) features a convex entropy with associated negative definite entropy production.
  $\longrightarrow$ **stability**

$\oplus$ stable

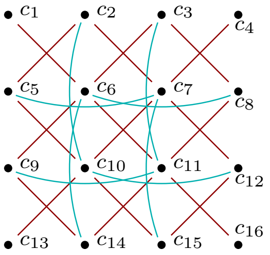
$\oplus$ accurate to 2^{nd} order.

Generalizations to **higher orders** possible.

16 Discrete Velocity Model [Babovsky, Kauf 2008]

Goal: Analyse abilities of different closures in simplified (1+1+2 dim) collision model.

Discrete velocity space:



Linearization and Fourier transform (in x):

$$\partial_t \hat{f}_j^k(t) - i \sum_{j=1}^{16} V_{ij} k \hat{f}_j^k(t) + \frac{1}{\varepsilon} \sum_{j=1}^{16} K_{ij} \hat{f}_j^k(t) = 0,$$

exact Solution:

$$\hat{\mathbf{f}}^k(t) = \exp[\overbrace{ik\mathbf{V}}^{\text{transport} \equiv \textbf{oscillation}} \underbrace{-\frac{1}{\varepsilon}\mathbf{K}}_{\text{interaction} \equiv \textbf{damping}}]\hat{\mathbf{f}}^k(0)$$

Features of K:

- dim$(\ker(K)) = 4$
- dim$(V_1) = 3$
- **dim**$(V_2) = 9$

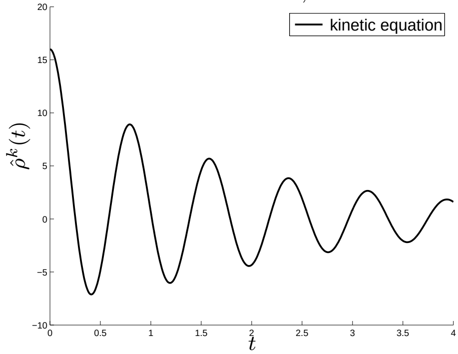
Low dimensional multiscale approximation of high dimensional problem.

4/16 (Euler, NSF) 7/16 (Grad, Scale Induced Closure)

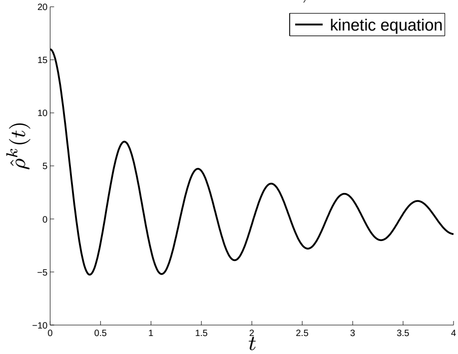
Comparison of Results

$$\hat{\rho}^k(t) = \sum_{j=1}^{16} \hat{f}_j^k(t)$$

$k = 3, \quad \varepsilon = \mathbf{0.1}$



$k = 3, \quad \varepsilon = \mathbf{0.5}$



transport ≡ **oscillation**

exact Solution: $\hat{\mathbf{f}}^k(t) = \exp[\overbrace{ik\mathbf{V}} \underbrace{-\frac{1}{\varepsilon}\mathbf{K}}]\hat{\mathbf{f}}^k(0)$

interaction ≡ **damping**

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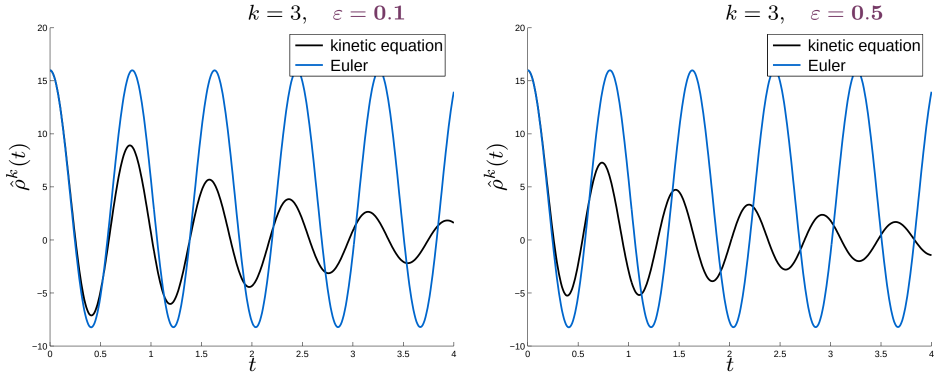

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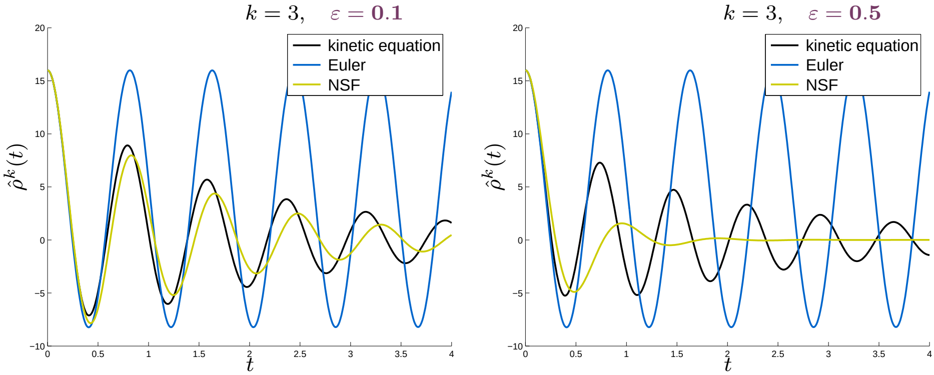

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exact Solution: $$\hat{\mathbf{f}}^k(t) = \exp[\ \overbrace{ik\mathbf{V}}\ \underbrace{-\frac{1}{\varepsilon}\mathbf{K}}\ ]\hat{\mathbf{f}}^k(0)$$

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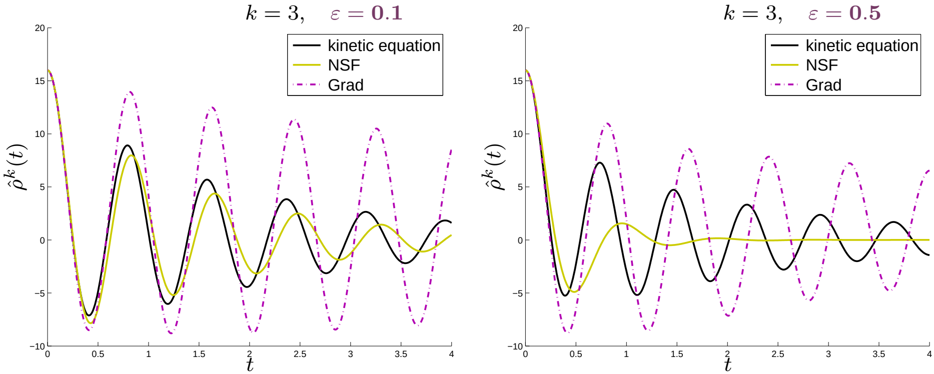

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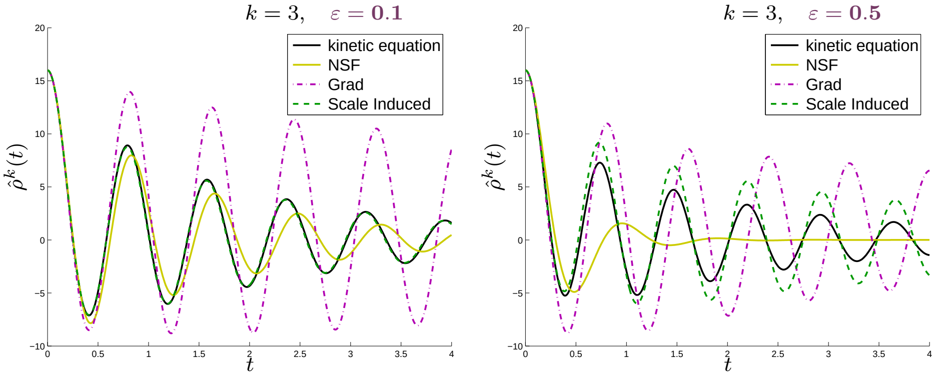

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transport ≡ **oscillation**

interaction ≡ **damping**

Beyond Kinetic Theory

Example from Kinetic Theory shows **more general abilities** of Scale Induced Closure:

$\mathbf{N \gg 1}$

$$y'(t) = Ay - \frac{1}{\varepsilon} By, \qquad y \in \mathbb{R}^N, \qquad B \text{ sym. pos. semidef.}$$

- Parametrize (low dimensional) $\ker B$ by ρ, M.
- Choose accuracy of order C $y = M\rho + \varepsilon G_1^{A,B} \mu_1^{A,B} + \dots + \varepsilon^C G_C^{A,B} \mu_C^{A,B}$
- **Order of Magnitude** determines higher moments $\mu_i^{A,B}$, $G_i^{A,B}$ out of A and B, $i = 1, \dots, C$.
- Solve low dimensional system for ρ, $\mu_1^{A,B}$,..., $\mu_C^{A,B}$

General approximation theory **without spectral gap** but with **cascade of scales** induced by $1, \varepsilon, \dots \varepsilon^C$.

Any Examples ?

Summary

- Kinetic Equation: high dimensional, many details
- Classical Closures:
  - Chapman-Enskog: $f = M\rho + \varepsilon f_1 + \varepsilon^2 f_2 + \mathcal{O}\left(\varepsilon^3\right)$; equilibrium variables used.
    $\oplus$ accurate, $\ominus$ higher orders unstable.
  - Grad: $f = M\rho + G\mu$; arbitrary direction in phase space.
    $\ominus$ unclear accuracy, $\oplus$ stable.
- Scale Induced Closure: $f = M\rho + \varepsilon G\mu + \mathcal{O}\left(\varepsilon^2\right)$,
  kinetic equation $\longrightarrow G = -K^\dagger \mathbf{c} M D^\dagger$
  $\oplus$ accurate, $\oplus$ stable.

- Provides equations in **transition regime** of kinetic theory.
- Can be used for **low dimensional approximations** of **general stiff problems**.

Hybrid Approach: Ideas

$$\underbrace{\partial_t f_{\mathbf{x},t}(\mathbf{c}) + c_k \partial_{x_k} f_{\mathbf{x},t}(\mathbf{c})}_{\text{advection}} = \overbrace{\frac{1}{\varepsilon}}^{\text{scale}} \underbrace{J(f,f)}_{\text{collisions}} \overset{\text{BGK}}{=} -\frac{1}{\varepsilon}\left(\frac{\rho}{\sqrt{2\pi\theta}} \exp\left(\frac{(c-v)^2}{2\theta}\right) - f\right)$$

Problems:

- $f(\mathbf{x}, t, \mathbf{c})$ expensive ($\mathbf{c} \in \mathbb{R}^3$)
- Too many details in most situations
- Non-Galilei invariant formulation (moving mesh)

Ideas:

- Use weak formulation, transform to Galilei-invariant form.
- Approximate $f_{\mathbf{x},t}(\mathbf{c})$ appropriately (moments, point values, Gaussians,...)
- Couple to conservation laws to guarantee conservation of ρ, v,θ.

Galilei Invariant, Scaled Formulation

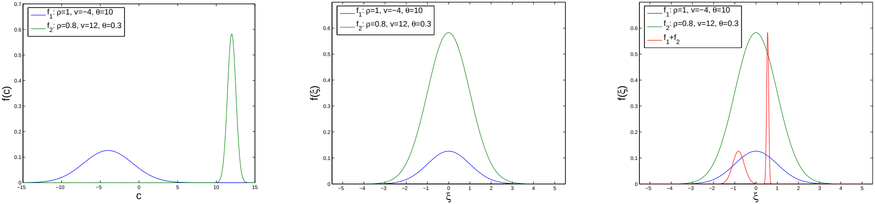
Transfer Equation (Boltzmann equation in weak form):

$$(\rho\langle\psi\rangle)_t + (\rho\langle c_k\psi\rangle)_{x_k} = m\int f\partial_t\psi dc + m\int f c_k \partial_{x_k}\psi dc + m\int \psi S(f,f)dc$$

Moving Mesh:

Rescale c_i to $\xi_i = \frac{c_i - v_i(\mathbf{x},t)}{\sqrt{\theta(\mathbf{x},t)}}$, writing $f(t,x,c) = \hat{\mathrm{f}}(t,x,\xi)$.

c-space | ξ-space | Bimodal in ξ-space



Galilei invariant! Domain **centered around zero**. No reduction in resolution.

Note: Cannot get full f from $\hat{\mathrm{f}}$ (unknown v,θ).

Linking constraints: $\int \xi\,\hat{\mathrm{f}}\,d\xi = 0,\quad \int \hat{\mathrm{f}}\,d\xi = \frac{\rho}{\sqrt{\theta}^d},\quad \frac{1}{d}\int \xi^2\,\hat{\mathrm{f}}\,d\xi = \frac{\rho}{\sqrt{\theta}^d}$

Decomposition of $\hat{\mathrm{f}}$

Moment Equations (no c) $\longleftarrow$ **?????** $\longrightarrow$ **pointwise discretization of c**

Moments offer highly **efficient** discretizations close to **equilibrium**, while **point values** capture **small scale** behaviour.

How to **combine**?

Decompose:

$$\hat{\mathrm{f}}(t,x,\xi) = \sum_{\beta=1}^{N} \kappa_\beta(x,t)\hat{\phi}_\beta(\xi)$$

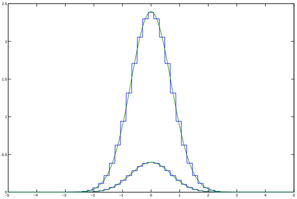
$\hat{\phi}_\alpha(\xi)$ **Basisfunctions**

$\kappa_\beta(t,x)$ **coefficients**.

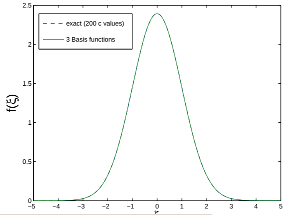
$\langle f,g\rangle := \int fg d\xi$ and $M^{\alpha\beta} = \langle\hat{\phi}_\alpha,\hat{\phi}_\beta\rangle$.

$$\Longrightarrow \kappa_\beta(t,x) = \sum_{\alpha=1}^{N} (M^{-1})_{\alpha\beta} \int \hat{\mathrm{f}}\,\hat{\phi}_\alpha d\xi$$

$\hat{\phi}$ Piecewise Constants



$\hat{\phi}$ Gaussian



Transfer Equation for $\kappa_\alpha(t,x)$

Rewrite transfer equation, testing with $\hat{\psi}(t,x,\xi) = \frac{1}{\rho(t,x)}\hat{\phi}(\xi)$:

$$(\rho\langle\psi\rangle)_t + (\rho\langle c_k\psi\rangle)_{x_k} = m\int f\partial_t\psi dc + m\int f c_k \partial_{x_k}\psi dc + m\int \psi S(f,f)dc$$

$\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow$ **'material equations'**

$$M^{\alpha\beta}\partial_t\kappa_\beta + \left[M^{\alpha\beta}v_k + M_k^{\alpha\beta}\sqrt{\theta}\right]\partial_{x_k}\kappa_\beta = \quad \Big\}\textbf{advec.}$$

$$\left.\begin{aligned}&+\left[\left(G_{s,s}^{\alpha\beta}\partial_t + \left(G_{s,s}^{\alpha\beta}v_k + G_{ks,s}^{\alpha\beta}\sqrt{\theta}\right)\partial_{\mathrm{x_k}}\right)\ln\left(\theta^{1/2}\right)\right.\\ &\left.+\frac{1}{\sqrt{\theta}}\left(G_{,j}^{\alpha\beta}\partial_t + (G_{,j}^{\alpha\beta}v_k + \sqrt{\theta}G_{k,j}^{\alpha\beta})\,\partial_{\mathrm{x_k}}\right)v_j - \frac{1}{\tau}M^{\alpha\beta}\right]\kappa_\beta(x,t)\end{aligned}\right\}\textbf{relax.}$$

$$+\frac{1}{\tau}\frac{\rho}{(2\pi\theta)^{d/2}}\int\hat{\phi}_\alpha(\xi)e^{-\xi_k^2}d\xi \quad \Big\}\textbf{source}$$

$$M_k^{\alpha\beta} = \int \xi_k\hat{\phi}_\alpha\hat{\phi}_\beta d\xi, \quad G_{,j}^{\alpha\beta} = \int\hat{\phi}_\alpha\,\partial_{\xi_j}\,\hat{\phi}_\beta d\xi, \quad G_{k,j}^{\alpha\beta} = \int\xi_k\hat{\phi}_\alpha\,\partial_{\xi_j}\,\hat{\phi}_\beta d\xi,$$

$$G_{kj,j}^{\alpha\beta} = \int\xi_k\xi_j\hat{\phi}_\alpha\,\partial_{\xi_j}\,\hat{\phi}_\beta d\xi$$

all pure numbers (indep. of t,x,ξ)

Coupling through Heat Flux

Main idea:

Solve **closure problem for q** in conservation laws by **coupling** to material equations:

$$
\begin{aligned}
q_k &= m \int (c_k - v_k)(c_l - v_l)^2 f dc = \sqrt{\theta}^{d+3} \int \xi^2 \xi_k \hat{\mathrm{f}} \, d\xi \\
&\cong m\sqrt{\theta}^{d+3} \sum_{\beta=1}^{N} \kappa_\beta(t,x) \int \xi^2 \xi_k \hat{\phi}_\beta d\xi := \sqrt{\theta}^{d+3} \sum_{\beta=1}^{N} \kappa_\beta V_{3;k}^{\beta}
\end{aligned}
$$

Define further **pure numbers**:

$$
\begin{aligned}
V_0^\alpha &:= \int \hat{\phi}_\alpha(\xi) d\xi \\
V_{1;i}^\alpha &:= \int \xi_i \hat{\phi}_\alpha(\xi) d\xi \\
V_2^\alpha &:= \int \xi_k \xi_k \hat{\phi}_\alpha(\xi) d\xi \\
V_{\exp}^\alpha &:= \int \hat{\phi}_\alpha(\xi) e^{-\xi_k^2} d\xi
\end{aligned}
$$

Features:

- Enable **macro-balance** through conservation laws.
- **Constraints couple** κ_β to macro-level.

Boltzmann ∪ Conservation Laws

Coupled equations in 1+1 dimensions

$$\left.\begin{aligned}&\partial_t\rho+\partial_x(\rho v)=0\\&\partial_t(\rho v)+\partial_x\left(\rho v^2+\rho\theta\right)=0\\&\partial_t\left(\tfrac{1}{2}\rho\theta+\tfrac{1}{2}\rho v^2\right)+\partial_x\left[\left(\tfrac{1}{2}\rho\theta+\tfrac{1}{2}\rho v^2\right)v+\rho\theta v+q\right]=0\end{aligned}\right\}\quad \textbf{cons. laws}$$

$$\left.\mathbf{q=m\theta^2\kappa_\beta(x,t)V^\beta_{3;k}}\right\}\quad \textbf{coupling}$$

$$\left.\begin{aligned}&M^{\alpha\beta}\partial_t\kappa_\beta+\left[M^{\alpha\beta}v_k+M_k^{\alpha\beta}\sqrt{\theta}\right]\partial_{x_k}\kappa_\beta=\\&\quad+\left[\left(G^{\alpha\beta}_{s,s}\partial_t+\left(G^{\alpha\beta}_{s,s}v_k+G^{\alpha\beta}_{ks,s}\sqrt{\theta}\right)\partial_{\mathrm{x_k}}\right)\ln\left(\theta^{1/2}\right)\right.\\&\quad\left.+\frac{1}{\sqrt{\theta}}\left(G^{\alpha\beta}_{,j}\partial_t+(G^{\alpha\beta}_{,j}v_k+\sqrt{\theta}G^{\alpha\beta}_{k,j})\,\partial_{\mathrm{x_k}}\right)v_j-\frac{1}{\tau}M^{\alpha\beta}\right]\kappa_\beta(x,t)\\&\quad+\frac{1}{\tau}\,\frac{\rho}{(2\pi\theta)^{d/2}}\int\hat{\phi}_\alpha(\xi)e^{-\xi_k^2}d\xi\end{aligned}\right\}\quad \textbf{mat. equations}$$

$$\left.\kappa_\beta(x,t)V_0^\beta\overset{!}{=}\kappa_\beta(x,t)V_2^\beta,\quad \kappa_\beta(x,t)V^\beta_{1;i}\overset{!}{=}0\right\}\quad \textbf{constraints}$$

Remarks:

- Use sum convention.
- Matrices M, G, vectors V, are **known pure numbers**.
- **Hyperbolicity?**, **accuracy?**
- **Constraints?**

Conclusions

- Scale Induced Closure: $f = M\rho + \varepsilon G\mu + \mathcal{O}\left(\varepsilon^2\right)$,
  kinetic equation $\longrightarrow G = -K^\dagger \mathbf{c} M D^\dagger$ $\oplus$ accurate, $\oplus$ stable.

  - Provides equations in **transition regime** of kinetic theory.
  - Can be used for **low dimensional approximations** of **general stiff problems**.

- Kinetic Equation: High detail resolution in all (kinetic) regimes.

  - **Galilei-invariant, scaled weak formulation**.
  - **General Decomposition** $\hat{\mathrm{f}} = \sum_{\beta=1}^{N} \kappa_\beta(x,t)\hat{\phi}_\beta(\xi)$
  - **Coupling** of Boltzmann and Conservation Laws through heat flux q.
  - **Hyperbolicity? Accuracy?**

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Engraziel fetg per Vossa attenziun !