Semi-Lagrangian Galerkin Methods for Discrete Differential Forms

Holger Heumann Ralf Hiptmair Jinchao Xu[1]

Seminar für Angewandte Mathematik, ETH Zürich
[1] Department of Mathematics, Pennsylvania State University,

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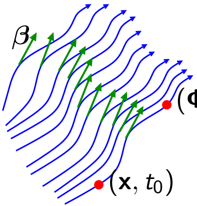
Scalar material derivatives

Flow field $\mathbf{\Phi}_{t_0} : \Omega \times [t_{min}, t_{max}] \mapsto \mathbb{R}^n$, s.t.:

$$\begin{aligned} \frac{d}{dt}\mathbf{\Phi}_{t_0}(\mathbf{x}, t) &= \boldsymbol{\beta}(\mathbf{\Phi}_{t_0}(\mathbf{x}, t), t) \\ \mathbf{\Phi}_{t_0}(\mathbf{x}, t_0) &= \mathbf{x} \end{aligned}$$



with velocity field

$$\boldsymbol{\beta} : \Omega \times [t_{min}, t_{max}] \mapsto \mathbb{R}^n, \, \Omega \subset \mathbb{R}^n$$

The total time derivative of a scalar function $u(\mathbf{\Phi}_{t_0}(\mathbf{x}, t), t)$:

$$\begin{aligned} \frac{d}{dt} u(\mathbf{\Phi}_{t_0}(\mathbf{x}, t), t)_{|_{t=t_0}} &= \boldsymbol{\beta}(\mathbf{x}, t_0) \cdot \mathbf{grad}\, u(\mathbf{x}, t_0) + \frac{\partial u(\mathbf{x}, t)}{\partial t}|_{t=t_0} \\ &= \lim_{\tau \to 0} \frac{u(\mathbf{x}, t_0) - u\left(\mathbf{\Phi}_{t_0}(\mathbf{x}, t_0 - \tau), t_0 - \tau\right)}{\tau} \end{aligned}$$

$$\boxed{\frac{d}{dt}\mathbf{\Phi}^*_{t_0,t} u(\mathbf{x}, t)_{|_{t=t_0}} \overset{Pullback}{=} \lim_{\tau \to 0} \frac{u(\mathbf{x}, t_0) - \mathbf{\Phi}^*_{t_0, t_0 - \tau} u(\mathbf{x}, t_0 - \tau)}{\tau}}$$

Differential Forms

Differential forms ω_p act on p dimensional manifolds M_p!

$$\omega_p(M_p) := \int_{M_p} \omega_p$$

Pullbacks leave the integrals invariant:

$$\int_{\Omega_p} \mathbf{\Phi}^*_{t,t_0} \omega_p = \int_{\mathbf{\Phi}_{t,t_0}(\Omega_p)} \omega_p$$

Examples in 3D:

- 0–forms $\leftrightarrow$ point evaluations :

$$\mathbf{\Phi}^*_{t,t_0} \omega_0 = \omega_0 \circ \mathbf{\Phi}_{t,t_0}$$

- 1–forms $\leftrightarrow$ line integrals:

$$\mathbf{\Phi}^*_{t,t_0} \omega_1 = \mathrm{D}_{\mathbf{x}}\, \mathbf{\Phi}^T_{t,t_0}\, \omega_1 \circ \mathbf{\Phi}_{t,t_0}$$

- 2–forms $\leftrightarrow$ surface integrals:

$$\mathbf{\Phi}^*_{t,t_0} \omega_2 = \det(\mathrm{D}_{\mathbf{x}}\, \mathbf{\Phi}_{t,t_0})\, \mathrm{D}_{\mathbf{x}}\, \mathbf{\Phi}^{-1}_{t,t_0}\, \omega_2 \circ \mathbf{\Phi}_{t,t_0}$$

- 3–forms $\leftrightarrow$ volume integrals:

$$\mathbf{\Phi}^*_{t,t_0} \omega_3 = \det(\mathrm{D}_{\mathbf{x}}\, \mathbf{\Phi}_{t,t_0}) \omega_3 \circ \mathbf{\Phi}_{t,t_0}$$

Material derivatives of forms

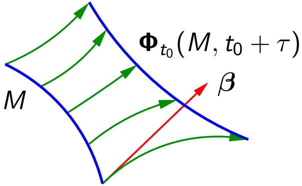
We consider the limit for differential p-forms ω:

$$\frac{d}{dt}\mathbf{\Phi}^*_{t_0,t}\omega(\mathbf{x},t)_{|t=t_0} = \lim_{\tau\to 0}\frac{\omega(\mathbf{x},t_0) - \mathbf{\Phi}^*_{t_0,t_0-\tau}\omega(\mathbf{x},t_0-\tau)}{\tau}$$

p-forms are linear forms on p-dim. manifolds ($\int_M \omega(x)\mathrm{d}x := \omega(M)$):

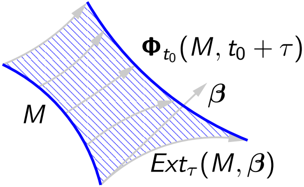
$$\frac{d}{dt}\omega(\mathbf{\Phi}_{t_0}(M,t),t)_{|t=t_0} =$$

$$\underbrace{\mathrm{d}\,\mathrm{i}_{\boldsymbol{\beta}}\,\omega(M,t_0) + \mathrm{i}_{\boldsymbol{\beta}}\,\mathrm{d}\,\omega(M,t_0)}_{\text{Lie-derivative } \mathrm{L}_{\boldsymbol{\beta}}} + \frac{\partial\omega(M,t)}{\partial t}|_{t=t_0}$$



d: exterior derivative
$\mathrm{i}_{\boldsymbol{\beta}}$: contraction/interior product

$$\mathrm{i}_{\boldsymbol{\beta}}\,\omega(M) := \lim_{\tau\to 0}\frac{\omega(Ext_\tau(M,\boldsymbol{\beta}))}{\tau}$$



1-form $\mathbf{a}$ in $\mathbb{R}^3$: $\mathrm{L}_{\boldsymbol{\beta}}\,\mathbf{a} = \mathbf{grad}(\boldsymbol{\beta}\cdot\mathbf{a}) - \boldsymbol{\beta}\times\mathbf{curl}\,\mathbf{a}$

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Semi-Lagrangian Galerkin Methods

Simple Euler step (ω: p-form, η: $n-p$-form)

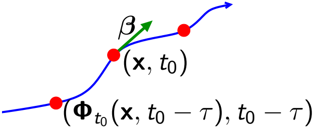
$$\int_\Omega \frac{d}{dt}\mathbf{\Phi}^*_{t_0,t}\omega(\mathbf{x},t)_{|_{t=t_0}} \wedge \eta(\mathbf{x})\ \mathrm{d}\mathbf{x} =$$

$$\lim_{\tau\to 0}\frac{1}{\tau}\left(\int_\Omega \omega(\mathbf{x},t_0)\wedge\eta(\mathbf{x})\ \mathrm{d}\mathbf{x} - \int_\Omega \mathbf{\Phi}^*_{t_0,t_0-\tau}\omega(\mathbf{x},t_0-\tau)\wedge\eta(\mathbf{x})\ \mathrm{d}\mathbf{x}\right)$$

Direct method:

$$\approx \frac{1}{\tau}\Big(\int_\Omega \omega(\mathbf{x},t_0)\wedge\eta(\mathbf{x})\ \mathrm{d}\mathbf{x} -$$

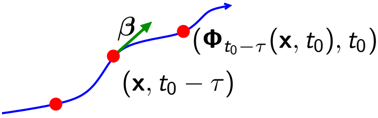
$$\int_\Omega \mathbf{\Phi}^*_{t_0,t_0-\tau}\omega(\mathbf{x},t_0-\tau)\wedge\eta(\mathbf{x})\ \mathrm{d}\mathbf{x}\Big)$$



Adjoint method: $(\mathbf{\Phi}_t(\Omega)=\Omega)$

$$\approx \frac{1}{\tau}\Big(\int_\Omega \omega(\mathbf{x},t_0)\wedge\eta(\mathbf{x})\ \mathrm{d}\mathbf{x} -$$

$$\int_\Omega \omega(\mathbf{x},t_0-\tau)\wedge\mathbf{\Phi}^*_{t_0-\tau,\tau}\eta(\mathbf{x})\ \mathrm{d}\mathbf{x}\Big)$$



Application: Eddy-current model in moving media

Reduced Maxwell's equation:

$$\begin{aligned} \mathrm{d}\, e &= -\partial_t b \\ \mathrm{d}\, h &= j \\ j &= *_\sigma (e + \mathrm{i}_{\boldsymbol{\beta}}\, b) \\ *_\mu h &= b \end{aligned} \quad \xrightarrow{\tilde{e} = e + \mathrm{i}_{\boldsymbol{\beta}}\, b,\ \mathrm{d}\, b = 0} \quad \begin{aligned} \mathrm{d}\, \tilde{e} &= \underbrace{-\partial_t b - \mathrm{d}\, \mathrm{i}_{\boldsymbol{\beta}}\, b - \mathrm{i}_{\boldsymbol{\beta}}\, \mathrm{d}\, b}_{\text{Material derivative}} \\ \mathrm{d}\, h &= j \\ j &= *_\sigma \tilde{e} \\ *_\mu h &= b \end{aligned}$$

Semi-discrete h-based formulation $\Rightarrow$ adjoint method:

$$\int_\Omega \mu\, h(\mathbf{x}, t) \wedge h'(\mathbf{x}) - \mu\, h(\mathbf{x}, \tau) \wedge \boldsymbol{\Phi}^*_{t-\tau,t} h'(\mathbf{x}) \mathrm{d}\mathbf{x}$$

$$= -\tau \int_\Omega \frac{1}{\sigma} \mathrm{d}\, h(\mathbf{x}, t) \wedge \mathrm{d}\, h'(\mathbf{x}) \mathrm{d}\mathbf{x}$$

Semi-discrete a-based formulation $\Rightarrow$ direct method: $(\mathrm{d}\, a = b)$

$$\int_\Omega \sigma\, a(\mathbf{x}, t) \wedge a'(\mathbf{x}) - \sigma\, \boldsymbol{\Phi}^*_{t,t-\tau} a(\mathbf{x}, \tau) \wedge a'(\mathbf{x}) \mathrm{d}\mathbf{x}$$

$$= -\tau \int_\Omega \frac{1}{\mu} \mathrm{d}\, a(\mathbf{x}, t) \wedge \mathrm{d}\, a'(\mathbf{x}) \mathrm{d}\mathbf{x}$$

Discrete Differential Forms

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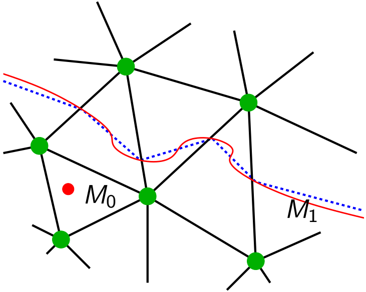
$\omega_p(M_p) := \int_{M_p} \omega_p$



Discrete setting:
prescribe ω_p on finitely many $M_p^{\mathbf{k}}$
(vertices $\mathbf{k} = i$, edges $\mathbf{k} = (e_1, e_2) \dots$).

Interpolation of $M_p \rightarrow$ approximation
$\omega_p(M_p) \cong \sum_{\mathbf{k}} a_{\mathbf{k}}(M_p)\, \omega_p(M_p^{\mathbf{k}})$

Limit procedure $\rightarrow$ Whitney forms $\omega_p^{\mathbf{k}}$

$$\omega_p(x) \cong \omega_p^h(x) = \sum_{\mathbf{k}} \omega_p^{\mathbf{k}}(x)\omega_p(M_p^{\mathbf{k}}), \quad \omega_p^{\mathbf{k}}(x) := \lim_{M_p \rightarrow x} a_{\mathbf{k}}(M_p)$$

- $p = 0$: ω_0^i Linear Finite Elements
- $p = 1$: $\omega_1^{\mathbf{e}}$ Edge Elements

$\Longrightarrow$ back in FEM-setting, but conforming!

Discretization strategies

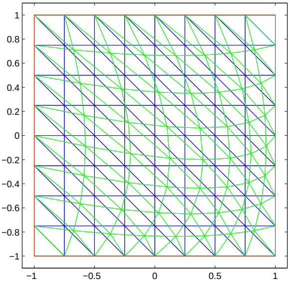
How to evaluate for e.g. lowest order Whitney 1-forms ω_N, η_N

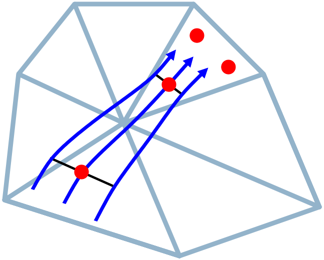
$$\int_\Omega \mathbf{\Phi}^*_{t_0, t_0-\tau} \omega_N(\mathbf{x}, t_0 - \tau) \wedge \eta_N(\mathbf{x}) \, d\mathbf{x} \quad ?$$

1.) Local quadrature based on point values at $\mathbf{x}_q$:

$$\mathbf{\Phi}^*_{t,t'} \omega_N(\mathbf{x}_q, t') = \mathsf{D}_\mathbf{x} \mathbf{\Phi}_t(\mathbf{x}_q, t')^T \omega_N(\mathbf{\Phi}_t(\mathbf{x}_q, t'), t')$$

with $\begin{cases} \frac{d}{dt} \mathbf{\Phi}_{t_0}(\mathbf{x}_q, t) = \boldsymbol{\beta}(\mathbf{\Phi}_{t_0}(\mathbf{x}_q, t), t); \quad \mathbf{\Phi}_{t_0}(\mathbf{x}_q, t_0) = \mathbf{x}_q \\ \frac{d}{dt} \mathsf{D}_\mathbf{x} \mathbf{\Phi}_{t_0}(\mathbf{x}_q, t) = \mathsf{D}_\mathbf{x} \boldsymbol{\beta}(\mathbf{\Phi}_{t_0}, t) \, \mathsf{D}_\mathbf{x} \mathbf{\Phi}_{t_0}(\mathbf{x}_q, t); \quad \mathsf{D}_\mathbf{x} \mathbf{\Phi}_{t_0}(\mathbf{x}_q, t_0) = \mathrm{Id} \end{cases}$





Discretization strategies

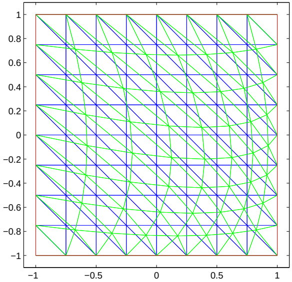
How to evaluate for e.g. lowest order Whitney 1-forms ω_N, η_N

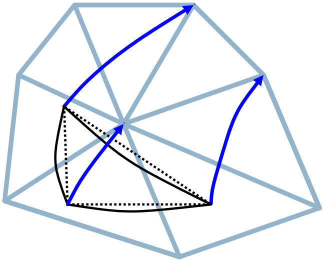
$$\int_\Omega \mathbf{\Phi}^*_{t_0,t_0-\tau}\omega_N(\mathbf{x}, t_0-\tau) \wedge \eta_N(\mathbf{x})\, \mathrm{d}\mathbf{x} \quad ?$$

2.) Edge interpolation:

$$\tilde{\omega}_N(\mathbf{x}, t') \;:=\; \Pi_N \mathbf{\Phi}^*_{t,t'}\omega_N(\mathbf{x}, t')$$

$$\tilde{\omega}_N(\mathbf{x}, t') \;=\; \sum_{\mathbf{e}} s_{\mathbf{e}}\,\omega_1^{\mathbf{e}} \quad \text{with} \quad s_{\mathbf{e}} = \int_{\mathbf{e}} \mathbf{\Phi}^*_{t,t'}\omega_N(\cdot, t')$$





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Discretization strategies

Important observation for $\mathbf{a}$ smooth form and $\mathbf{a}_N$ Whitney form:

$$\begin{aligned} L_{\boldsymbol{\beta}}\mathbf{a} = \mathbf{grad}(\boldsymbol{\beta}\cdot\mathbf{a}) - \boldsymbol{\beta}\times\mathbf{curl}\,\mathbf{a} &= \mathrm{D}\,\boldsymbol{\beta}^T\mathbf{a} + \mathrm{D}\,\mathbf{a}\boldsymbol{\beta} \\ (\mathrm{D}\,\mathbf{a}_N\boldsymbol{\beta})_{|T} &= (-\frac{1}{2}\boldsymbol{\beta}\times\mathbf{curl}\,\mathbf{a}_N)_{|T} \end{aligned}$$

Consequence:

- Any discretization of the Lie-derivative $L_{\boldsymbol{\beta}}$ based on point-values is at least consistent with $\mathrm{D}\,\boldsymbol{\beta}^T\mathbf{a} - \frac{1}{2}\boldsymbol{\beta}\times\mathbf{curl}\,\mathbf{a}$.
- Presumable failure of Semi-Lagrange based on quadrature.

Discretization strategies

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- Presumable failure of Semi-Lagrange based on quadrature.

Numerical experiment: Eddy current problem

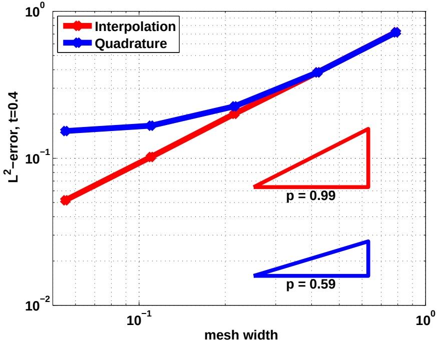
- unstructured, tetrahedral mesh on $[-1,1]^2$,
- $\mathbf{a}(\mathbf{x},t) = \cos(\pi t)(\sin(\pi x_1)(1-x_2),(1-x_1^2)(1-x_2^2))^T$,
- $\boldsymbol{\beta} = 0.66((1-x_1^2)(1-x_2^2),\sin(\pi x_1)\sin(\pi x_2))^T$,
- $\mathbf{f}(\mathbf{x},t) := \partial_t\mathbf{a}(\mathbf{x},t) + \mathrm{D}\,\boldsymbol{\beta}^T\mathbf{a} + \mathrm{D}\,\mathbf{a}\boldsymbol{\beta} + \mathbf{curl}\,\mathbf{curl}\,\mathbf{a}$,
- $\mathbf{f}_{mod}(\mathbf{x},t) := \partial_t\mathbf{a}(\mathbf{x},t) + \mathrm{D}\,\boldsymbol{\beta}^T\mathbf{a} - \frac{1}{2}\boldsymbol{\beta}\times\mathbf{curl}\,\mathbf{a} + \mathbf{curl}\,\mathbf{curl}\,\mathbf{a}$,

Discretization strategies

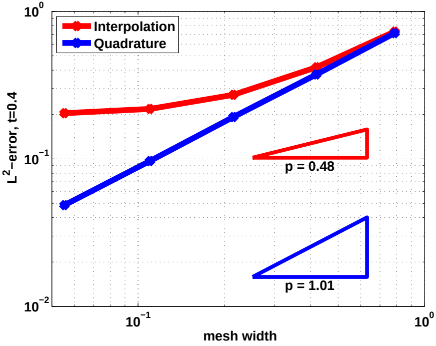
Discrete evolution for:

$$\partial_t \mathbf{a}(\mathbf{x}, t) + \mathbf{grad}(\boldsymbol{\beta} \cdot \mathbf{a}) - \boldsymbol{\beta} \times \mathbf{curl}\,\mathbf{a} + \mathbf{curl}\,\mathbf{curl}\,\mathbf{a} = \begin{cases} \mathbf{f} & \text{left figure} \\ \mathbf{f}_{mod} & \text{right figure} \end{cases}$$

$\mathbf{f}$ with complete Lie derivative:



$\mathbf{f}_{mod}$ with incomplete Lie derivative:



- Quadrature-based Semi-Lagrange solves different problem!
- Interpolation-based Semi-Lagrange is $O(h + \Delta t)$.

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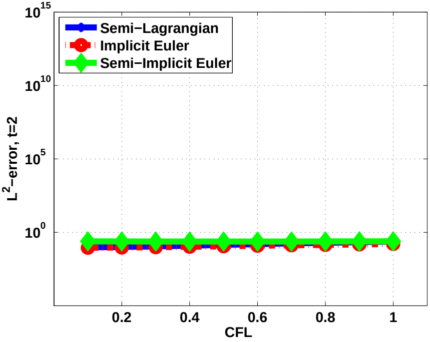
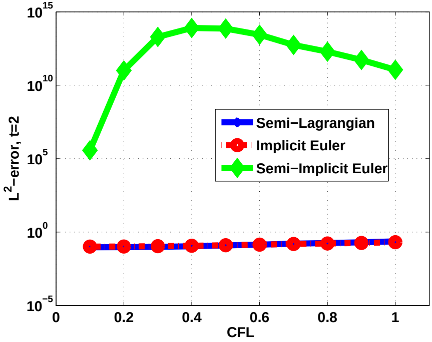
Convection-Diffusion of 1-forms: Stability

$\epsilon = 1$



$\epsilon = 10^{-5}$



- Semi-Lagrange: $(\mathbf{M} + \epsilon\Delta t\mathbf{C})\,\mathbf{a}^{k+1} = \mathbf{MP}\,\mathbf{a}^k + \Delta t(\mathbf{f}_1 + \mathbf{f}_2)$
- Implicit Euler: $(\mathbf{M} + \epsilon\Delta t\mathbf{C} + \Delta t\mathbf{L}_2)\,\mathbf{a}^{k+1} = \mathbf{M}\mathbf{a}^k + \Delta t\mathbf{f}_2$
- Semi-Implicit Euler: $(\mathbf{M} + \epsilon\Delta t\mathbf{C})\,\mathbf{a}^{k+1} = (\mathbf{M} - \Delta t\mathbf{L}_2)\,\mathbf{a}^k + \Delta t\mathbf{f}_2$

$\mathbf{P}$: nodal interpolator of pullback

$\mathbf{L}_2$: stiffness matrix for $\int_\Omega \boldsymbol{\beta} \times \mathbf{curl}\,\mathbf{A} \wedge \mathbf{A}'\mathrm{d}x$

Conclusions and Further Issues

Conclusions

- Semi-Lagrange formulation based on
  - nodal interpolation works
  - quadrature fails
- Constraint $\operatorname{div} \mathbf{b} = 0$ in Eddy current:
  - strong preservation in **a**-formulation ($\mathbf{curl}\,\mathbf{a} = \mathbf{b}$)
  - weak preservation in **h**-formulation ($\mathrm{d}\,\Pi_N \mathbf{\Phi}^* = \Pi_N \mathbf{\Phi}^*\,\mathrm{d}$)

Further Issues

- convergence theory
- stability
- $\mathbf{\Phi}(\Omega) \neq \Omega$
- generalization to higher order ansatz spaces
- piecewise polynomial approximation of flow of edges
- 3D applications