Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Foundations of the stochastic Galerkin method

Claude Jeffrey Gittelson

ETH Zurich, Seminar for Applied Mathematics

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Stochastic diffusion equation

$D \subset \mathbb{R}^d$ Lipschitz, for $\omega \in \Omega$,

$$-\nabla \cdot (a(\omega, x)\nabla u(\omega, x)) = f(x) \qquad x \in D ,$$
$$u(\omega, x) = 0 \qquad x \in \partial D .$$

Probability space $(\Omega, \mathscr{A}, P)$.

Parametric weak formulation: For $\omega \in \Omega$, find $u(\omega) \in H_0^1(D)$ s.t.

$$\int_D a(\omega, x)\nabla u(\omega, x) \cdot \nabla v(x)\mathrm{d}x = \int_D f(x)v(x)\mathrm{d}x$$

for all $v \in H_0^1(D)$.

Outline

stochastic problem on	$\Omega \to H_0^1(D)$
transform parameter domain	$\mathbb{R}^{\mathbb{N}} \to H_0^1(D)$
weak formulation	$L_\rho^2(\mathbb{R}^{\mathbb{N}}; H_0^1(D))$
stochastic basis	$\ell^2(\Lambda; H_0^1(D))$
finite element approximation	$\mathscr{V}_{\boldsymbol{j}}$

Transformation to a deterministic problem

Random variable $Y = (Y_m)_{m\in\mathbb{N}} : \Omega \to \mathbb{R}^{\mathbb{N}}$,

$$a(\omega, x) = a_Y(Y(\omega), x) \qquad \forall \omega \in \Omega \, .$$

Assume $(Y_m)_{m\in\mathbb{N}}$ independent.

$$\rho := Y(P) = \bigotimes_{m=1}^{\infty} \rho_m \; , \qquad \rho_m := Y_m(P)$$

Then $u(\omega) = u_Y(Y(\omega))$ for all $\omega \in \Omega$, where $u_Y(y) \in H_0^1(D)$ solves

$$\int_D a_Y(y, x) \nabla u_Y(y, x) \cdot \nabla v(x) \mathrm{d}x = \int_D f(x) v(x) \mathrm{d}x$$

for all $v \in H_0^1(D)$ and all $y \in \mathbb{R}^{\mathbb{N}}$.

Example: Karhunen-Loève expansion

$$a(\omega, x) = a_0(x) + \sum_{m=1}^{\infty} \sqrt{\lambda_m}\, \varphi_m(x) Y_m(\omega)\ .$$

- $(\varphi_m)_{m\in\mathbb{N}}$ are eigenfunctions of the covariance operator of $a(\cdot,\cdot)$ with eigenvalues λ_m.
- $(Y_m)_{m\in\mathbb{N}}$ are uncorrelated random variables.

Define

$$a_Y(y, x) := a_0(x) + \sum_{m=1}^{\infty} \sqrt{\lambda_m}\, \varphi_m(x) y_m\ .$$

Then $a(\omega, x) = a_Y(Y(\omega), x)$.

Example: Karhunen-Loève expansion

Alternative: Karhunen-Loève expansion of $\log(a(\omega, x))$,

$$a(\omega, x) = a_0(x) \exp\left(\sum_{m=1}^{\infty} \sqrt{\lambda_m}\, \varphi_m(x) Y_m(\omega)\right)$$
$$= a_0(x) \prod_{m=1}^{\infty} \exp\left(\sqrt{\lambda_m}\, \varphi_m(x) Y_m(\omega)\right) .$$

Define

$$a_Y(y, x) := a_0(x) \exp\left(\sum_{m=1}^{\infty} \sqrt{\lambda_m}\, \varphi_m(x) y_m\right) .$$

Then $a(\omega, x) = a_Y(Y(\omega), x)$.

Weak formulation

For all $y \in \mathbb{R}^{\mathbb{N}}$,Find $u_Y \in L^2_\rho(\mathbb{R}^{\mathbb{N}}; H^1_0(D))$ such that

$$\int_{\mathbb{R}^{\mathbb{N}}} \int_D a_Y(y) \nabla u_Y(y) \cdot \nabla v(y) \mathrm{d}x \mathrm{d}\rho(y) = \int_{\mathbb{R}^{\mathbb{N}}} \int_D f v(y) \mathrm{d}x \mathrm{d}\rho(y)$$

for $v \in H^1_0(D)$.for all $v \in L^2_\rho(\mathbb{R}^{\mathbb{N}}; H^1_0(D))$.

Product structure: $L^2_\rho(\mathbb{R}^{\mathbb{N}}; H^1_0(D)) \cong L^2_\rho(\mathbb{R}^{\mathbb{N}}) \otimes H^1_0(D)$.

- deterministic space $H^1_0(D)$,
- stochastic space $L^2_\rho(\mathbb{R}^{\mathbb{N}})$.

Weak formulation

Assume $f \in H^{-1}(D)$ and $a_Y(\cdot,\cdot)$ is uniformly bounded,

$$0 < \underline{a} \leq a_Y(y,x) \leq \overline{a} < \infty \qquad \forall x \in D\,, \quad \forall y \in \mathbb{R}^{\mathbb{N}}\,.$$

Theorem

u_Y is the unique element of $L^2_\rho(\mathbb{R}^{\mathbb{N}}; H^1_0(D))$ such that

$$\int_{\mathbb{R}^{\mathbb{N}}} \int_D a_Y \nabla u_Y \cdot \nabla v \, \mathrm{d}x \mathrm{d}\rho = \int_{\mathbb{R}^{\mathbb{N}}} \int_D f v \, \mathrm{d}x \mathrm{d}\rho$$

for all $v \in L^2_\rho(\mathbb{R}^{\mathbb{N}}; H^1_0(D))$. Furthermore,

$$\|u_Y\|_{L^\infty_\rho(\mathbb{R}^{\mathbb{N}}; H^1_0(D))} \leq \frac{1}{\underline{a}} \|f\|_{H^{-1}(D)}\,.$$

Stochastic semi-discretization

Let $(\varphi_\mu)_{\mu\in\Lambda}$ be an orthonormal basis of $L^2_\rho(\mathbb{R}^\mathbb{N})$ and

$$u_Y(y,x) = \sum_{\mu\in\Lambda} u_\mu(x)\varphi_\mu(y) \ , \quad (u_\mu)_{\mu\in\Lambda} \in \ell^2\big(\Lambda; H^1_0(D)\big) \ .$$

Define $\quad a_{\nu\mu}(x) := \int_{\mathbb{R}^\mathbb{N}} a_Y(y,x)\varphi_\mu(y)\varphi_\nu(y)\mathrm{d}\rho(y) \ , \quad \mu,\nu\in\Lambda \ ,$

and $\quad f_\nu(x) := f(x)\int_{\mathbb{R}^\mathbb{N}} \varphi_\nu(y)\mathrm{d}\rho(y) \ , \quad \nu\in\Lambda \ .$

Then using the test function $v(x)\varphi_\nu(y)$ for $v\in H^1_0(D)$, $\nu\in\Lambda$,

$$\sum_{\mu\in\Lambda}\int_D a_{\nu\mu}(x)\nabla u_\mu(x)\cdot\nabla v(x)\mathrm{d}x = \int_D f_\nu(x)v(x)\mathrm{d}x \ .$$

System of deterministic equations for the coefficients of u_Y.

Stochastic semi-discretization

By Parseval's identity,

$$\|u_Y\|_{L^2_\rho(\mathbb{R}^\mathbb{N};H^1_0(D))} = \|(u_\mu)_{\mu\in\Lambda}\|_{\ell^2(\Lambda;H^1_0(D))} \ .$$

Theorem

$u_Y \in L^2_\rho(\mathbb{R}^\mathbb{N}; H^1_0(D))$ *satisfies*

$$\int_{\mathbb{R}^\mathbb{N}} \int_D a_Y \nabla u_Y \cdot \nabla v \mathrm{d}x \mathrm{d}\rho = \int_{\mathbb{R}^\mathbb{N}} \int_D f v \mathrm{d}x \mathrm{d}\rho \quad \forall v \in L^2_\rho\Big(\mathbb{R}^\mathbb{N}; H^1_0(D)\Big)$$

if and only if $(u_\mu)_{\mu\in\Lambda} \in \ell^2(\Lambda; H^1_0(D))$ *satisfies*

$$\sum_{\mu\in\Lambda} \int_D a_{\nu\mu} \nabla u_\mu \cdot \nabla v \, \mathrm{d}x = \int_D f_\nu v \, \mathrm{d}x \quad \forall v \in H^1_0(D) \ , \ \nu \in \Lambda \ .$$

Tensor-product stochastic basis

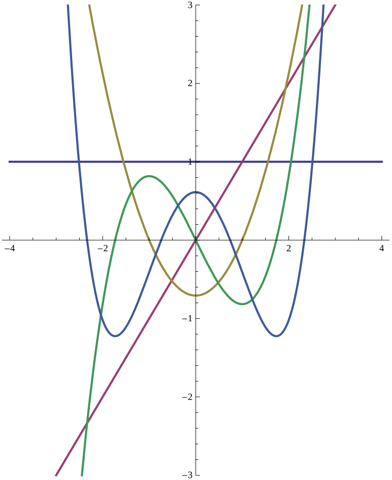
Construct orthonormal basis $(\varphi_\mu)_{\mu \in \Lambda}$ of $L^2_\rho(\mathbb{R}^\mathbb{N})$ with elements of the form

$$\varphi_\mu(y) = \varphi^{m_1}_{\nu_1}(y_{m_1}) \cdot \ldots \cdot \varphi^{m_k}_{\nu_k}(y_{m_k}) \ .$$

$(\varphi^m_\nu)_{\nu \in \Lambda_m}$ is an orthonormal basis of $L^2_{\rho_m}(\mathbb{R})$, such as

- polynomials
- piecewise polynomials
- wavelets

Example: orthonormal polynomials



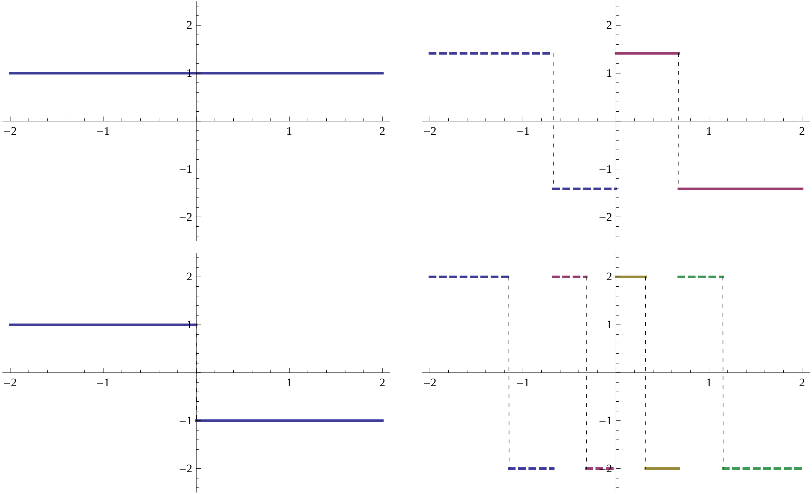
$(P_n^m)_{n\in\mathbb{N}_0}$ orthonormal polynomial basis of $L^2_{\rho_m}(\mathbb{R})$, $P_0^m = 1$, $\deg P_n^m = n$.

$$\Lambda := \ell^1(\mathbb{N};\mathbb{N}_0)$$

$$P_\mu := \bigotimes_{m=1}^{\infty} P^m_{\mu_m}, \qquad \mu \in \Lambda$$

$$\left(P_\mu(y) = \prod_{m\in\operatorname{supp}\mu}^{\infty} P^m_{\mu_m}(y_m) \right)$$

Example: Haar wavelets



General tensor-product construction

Let $(\varphi_\nu^m)_{\nu\in\Lambda_m}$ be an orthonormal basis of $L^2_{\rho_m}(\mathbb{R})$ with $\varphi_0^m = 1$. Then

$$(\varphi_\nu^m)_{\nu\in\Delta_m}\ , \qquad \Delta_m := \Lambda_m \setminus \{0\}$$

is an orthonormal basis of $L^2_{\rho_m}(\mathbb{R})/\mathbb{R}$.

For $I \in \mathscr{F}(\mathbb{N})$ a finite subset of $\mathbb{N}$,

$$\varphi_\mu := \bigotimes_{m\in I} \varphi^m_{\mu_m}\ , \qquad \mu \in \Delta_I := \prod_{m\in I} \Delta_m\ .$$

Define $W_I := \operatorname{span}\{\varphi_\mu\ ;\ \mu \in \Delta_I\} \subset L^2_\rho(\mathbb{R}^\mathbb{N})$. Then

$$\bigotimes_{m\in I} L^2_{\rho_m}(\mathbb{R}) = \bigoplus_{J\subset I} W_J = \bigoplus_{k=0}^{\#I} \bigoplus_{\substack{J\subset I\\ \#J=k}} W_J\ .$$

General tensor-product construction

Theorem

$$(\varphi_\mu)_{\mu \in \Lambda} , \quad \Lambda := \bigsqcup_{I \in \mathscr{F}(\mathbb{N})} \Delta_I ,$$

is an orthonormal basis of $L^2_\rho(\mathbb{R}^\mathbb{N})$*.*

Proof.

$(\varphi_\mu)_{\mu \in \Delta_I}$ is an orthonormal basis of W_I for all $I \in \mathscr{F}(\mathbb{N})$ and

$$L^2_\rho\left(\mathbb{R}^\mathbb{N}\right) = \bigoplus_{I \in \mathscr{F}(\mathbb{N})} W_I = \bigoplus_{k=0}^{\infty} \bigoplus_{\substack{I \subset \mathbb{N} \\ \#I = k}} W_I .$$

□

Finite element approximation

Define nested finite element spaces

$$\{0\} = V_0 \subset V_1 \subset \ldots \subset V_j \subset V_{j+1} \subset \ldots \subset H_0^1(D) \ .$$

For a sequence $\boldsymbol{j} = (j(\mu))_{\mu\in\Lambda} \in \ell^1(\Lambda;\mathbb{N}_0)$, define

$$\mathscr{V}_{\boldsymbol{j}} := \left\{ v = \sum_{\mu\in\Lambda} v_\mu \otimes \varphi_\mu \boldsymbol{v} = (v_\mu)_{\mu\in\Lambda} \ ; \ v_\mu \in V_{j(\mu)} \ \forall \mu \in \Lambda \right\} \subset L^2_\rho\Big(\mathbb{R}^{\mathbb{N}}; H$$

The Galerkin projection $\widetilde{u} \in \mathscr{V}_{\boldsymbol{j}}$ is determined by

$$\sum_{\mu\in\Lambda} \int_D a_{\nu\mu}(x) \nabla \widetilde{u}_\mu(x) \cdot \nabla \widetilde{v}(x) \mathrm{d}x = \int_D f_\nu(x) \widetilde{v}(x) \mathrm{d}x$$

for all $\widetilde{v} \in V_{j(\nu)}$, $\nu \in \Lambda$.

Finite element approximation

Theorem

There is a unique $\widetilde{u} \in \mathscr{V}_{\boldsymbol{j}}$ *such that*

$$\sum_{\mu \in \Lambda} \int_D a_{\nu\mu} \nabla \widetilde{u}_\mu \cdot \nabla \widetilde{v} \, \mathrm{d}x = \int_D f_\nu \widetilde{v} \, \mathrm{d}x \quad \forall \widetilde{v} \in V_{j(\nu)} \; , \; \nu \in \Lambda \; .$$

It satisfies the bound

$$\|\widetilde{u}\|_{L^2_\rho(\mathbb{R}^{\mathbb{N}};H^1_0(D))} \leq \frac{1}{\underline{a}} \|f\|_{H^{-1}(D)}$$

and is quasi-optimal,

$$\|\widetilde{u} - u_Y\|_{L^2_\rho(\mathbb{R}^{\mathbb{N}};H^1_0(D))} \leq \sqrt{\frac{\overline{a}}{\underline{a}}} \inf_{\widetilde{w} \in \mathscr{V}_{\boldsymbol{j}}} \|\widetilde{w} - u_Y\|_{L^2_\rho(\mathbb{R}^{\mathbb{N}};H^1_0(D))} \; .$$

Summary

We

- constructed tensor-product orthonormal bases of $L^2_\rho(\mathbb{R}^\mathbb{N})$,
- and used these to recast a stochastic boundary value problem as an infinite system of deterministic equations,
- which can be solved by Galerkin approximation using standard finite element spaces.

Thank you for your attention!