Entropy-stable discontinuous Galerkin finite element method with streamline diffusion and shock-capturing

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Goal

Find a numerical scheme for conservation laws

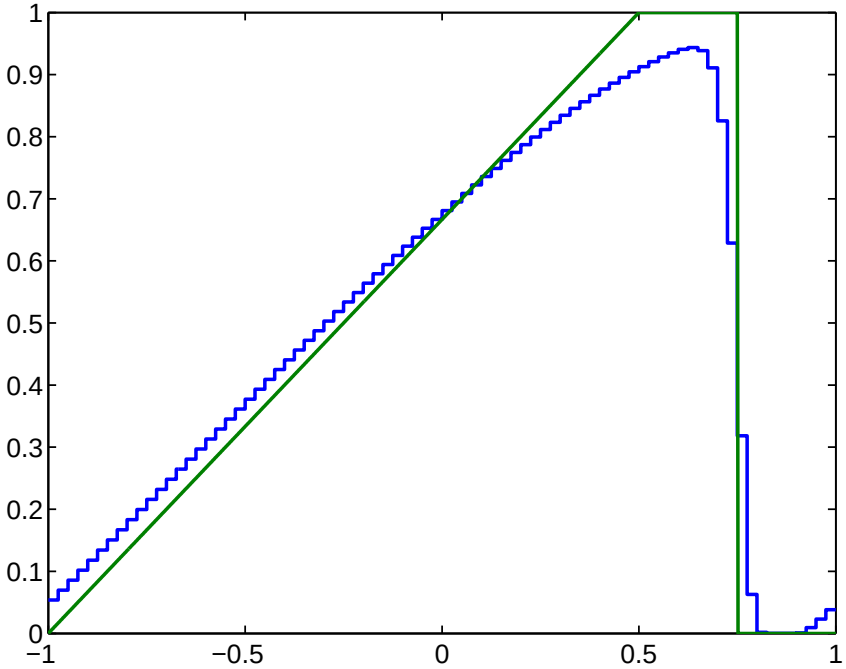
$$\boldsymbol{u}_t + \sum_{i=1}^{d} \boldsymbol{f}^i(\boldsymbol{u})_{x_i} = 0$$

which

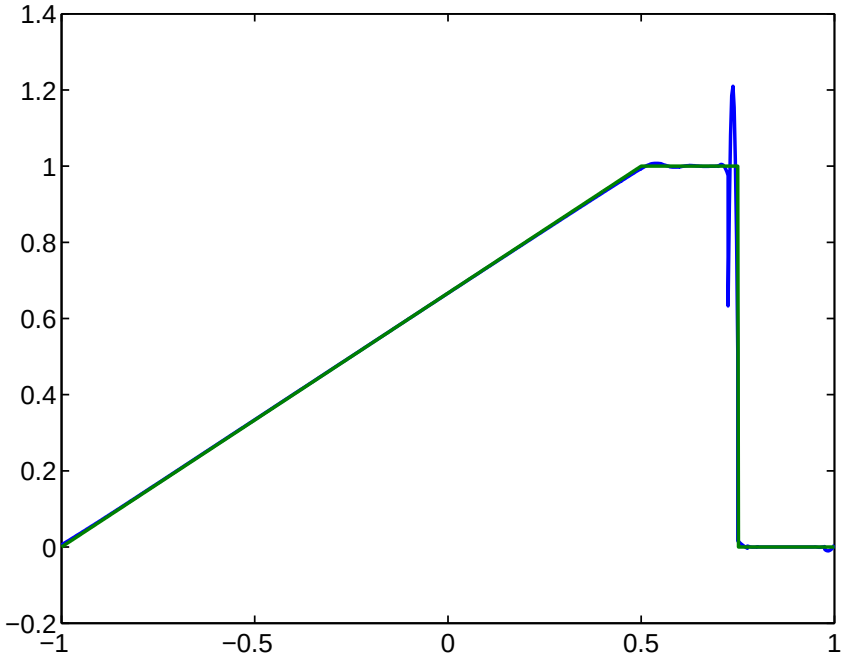
- is arbitrarily high-order accurate
- is entropy stable
- satisfies a maximum principle (bound in L^∞)
- converges for scalar conservation laws
- is multidimensional
- is reasonably efficient

Avoid oscillations and too much diffusion

neither

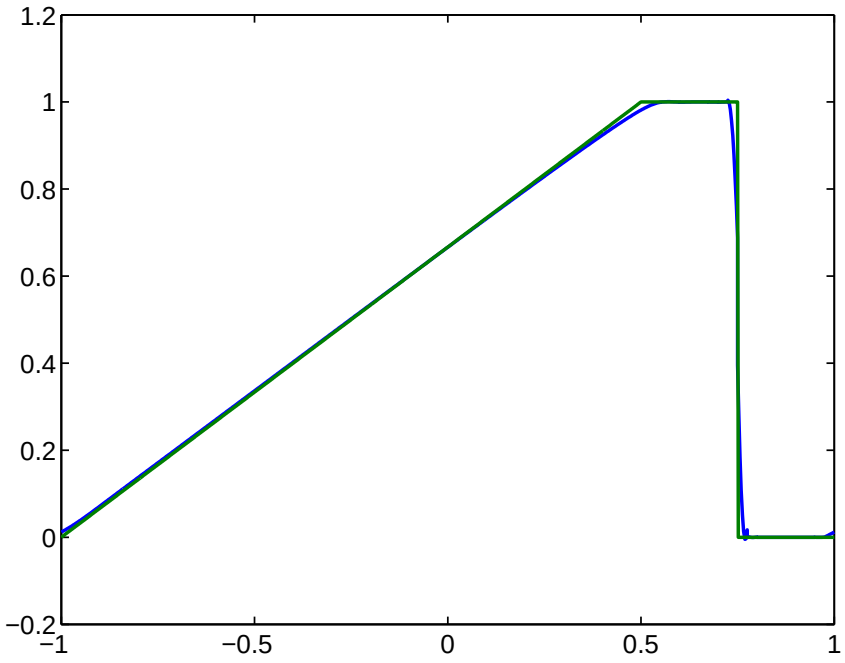


nor



Avoid oscillations and too much diffusion II

but rather



Outline

Derivation of the entropy stable DG FEM

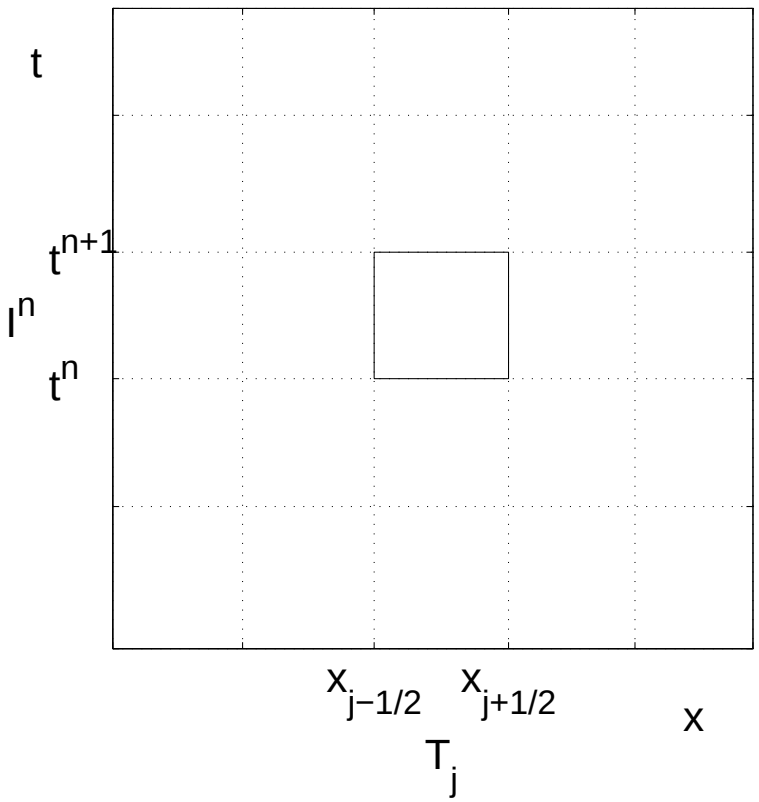
Start with the conservation law

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0$$

Multiply with a test function $\boldsymbol{w}$ (smooth)

$$\boldsymbol{u}_t \cdot \boldsymbol{w} + \boldsymbol{f}(\boldsymbol{u})_x \cdot \boldsymbol{w} = 0$$

Derivation of the entropy stable DG FEM II



Integrate over the elements

$$\sum_{n=0}^{N-1} \sum_{j \in J} \int_{I^n} \int_{T_j} (\boldsymbol{u}_t \cdot \boldsymbol{w} + \boldsymbol{f}(\boldsymbol{u})_x \cdot \boldsymbol{w}) \, \mathrm{d}x\mathrm{d}t = 0$$

Derivation of the entropy stable DG FEM III

Integrate by parts

$$\sum_{n=0}^{N-1}\sum_{j\in J}\Bigg(-\int_{I^n}\int_{T_j}(\boldsymbol{u}\cdot\boldsymbol{w}_t+\boldsymbol{f}(\boldsymbol{u})\cdot\boldsymbol{w}_x)\,\mathrm{d}x\mathrm{d}t$$
$$+\int_{T_j}\boldsymbol{u}(t_-^{n+1})\cdot\boldsymbol{w}(t_-^{n+1})\mathrm{d}x-\int_{T_j}\boldsymbol{u}(t_+^{n})\cdot\boldsymbol{w}(t_+^{n})\mathrm{d}x$$
$$+\int_{I^n}\boldsymbol{f}\left(\boldsymbol{u}_{j+1/2}^-\right)\cdot\boldsymbol{w}(x_{j+1/2}^-)\mathrm{d}t-\int_{I^n}\boldsymbol{f}\left(\boldsymbol{u}_{j-1/2}^+\right)\cdot\boldsymbol{w}(x_{j-1/2}^+)\mathrm{d}t\Bigg)=0$$

Numerical fluxes

Replace the flux at the boundary by numerical fluxes that depend on states on both sides of the boundary

$$\sum_{n=0}^{N-1}\sum_{j\in J}\Bigg(-\int_{I^n}\int_{T_j}(\boldsymbol{u}\cdot\boldsymbol{w}_t+\boldsymbol{f}(\boldsymbol{u})\cdot\boldsymbol{w}_x)\,\mathrm{d}x\mathrm{d}t$$
$$+\int_{T_j}\boldsymbol{U}\big(\boldsymbol{u}(t_-^{n+1}),\boldsymbol{u}(t_+^{n+1})\big)\cdot\boldsymbol{w}(t_-^{n+1})\mathrm{d}x-\int_{T_j}\boldsymbol{U}\big(\boldsymbol{u}(t_-^{n}),\boldsymbol{u}(t_+^{n})\big)\cdot\boldsymbol{w}(t_+^{n})\mathrm{d}x$$
$$+\int_{I^n}\boldsymbol{F}\Big(\boldsymbol{u}_{j+1/2}^-,\boldsymbol{u}_{j+1/2}^+\Big)\cdot\boldsymbol{w}(x_{j+1/2}^-)\mathrm{d}t$$
$$-\int_{I^n}\boldsymbol{F}\Big(\boldsymbol{u}_{j-1/2}^-,\boldsymbol{u}_{j-1/2}^+\Big)\cdot\boldsymbol{w}(x_{j-1/2}^+)\mathrm{d}t\Bigg)=0$$

For the numerical flux $\boldsymbol{U}$ we use the upwind flux,

$$\boldsymbol{U}\big(\boldsymbol{u}(t_-^n),\boldsymbol{u}(t_+^n)\big)=\boldsymbol{u}(t_-^n)$$

only this allows to do time stepping

Entropy stability

Choose entropy function $S(\boldsymbol{u})$ and an associated flux $Q(\boldsymbol{u})$
Want a discrete analogon of the entropy inequality

$$S_t + Q_x \leq 0$$

1. variable transformation: (entropy symmetrisation)

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{v})$$

   where $\boldsymbol{v} = S_{\boldsymbol{u}}$ are the entropy variables.
   Discretize $\boldsymbol{v}$ instead of $\boldsymbol{u}$.

2. entropy stable numerical flux: ([Tad87])

$$\boldsymbol{F}(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{F}^*(\boldsymbol{a}, \boldsymbol{b}) - \frac{\boldsymbol{D}}{2}(\boldsymbol{v}(\boldsymbol{b}) - \boldsymbol{v}(\boldsymbol{a}))$$

   $\boldsymbol{F}^*$: an entropy conservative flux.
   It has to fulfil a certain condition ($\Rightarrow$ unique in scalar case).
   $\boldsymbol{D}$: Diffusion matrix. Can be anything, as long as it is positive semidefinite.

Complete description

Choose the space of ansatz functions V (piecewise polynomials).
Then the description is complete:
Find $\boldsymbol{v}$ in V, such that

$$\sum_{n=0}^{N-1}\sum_{j\in J}\Bigg(-\int_{I^n}\int_{T_j}(\boldsymbol{u}(\boldsymbol{v})\cdot\boldsymbol{w}_t+\boldsymbol{f}(\boldsymbol{u}(\boldsymbol{v}))\cdot\boldsymbol{w}_x)\,\mathrm{d}x\mathrm{d}t$$
$$+\int_{T_j}\boldsymbol{u}(\boldsymbol{v}(t_-^{n+1}))\cdot\boldsymbol{w}(t_-^{n+1})\mathrm{d}x-\int_{T_j}\boldsymbol{u}(\boldsymbol{v}(t_-^{n}))\cdot\boldsymbol{w}(t_+^{n})\mathrm{d}x$$
$$+\int_{I^n}\boldsymbol{F}\Big(\boldsymbol{u}(\boldsymbol{v}_{j+1/2}^-),\boldsymbol{u}(\boldsymbol{v}_{j+1/2}^+)\Big)\cdot\boldsymbol{w}(x_{j+1/2}^-)\mathrm{d}t$$
$$-\int_{I^n}\boldsymbol{F}\Big(\boldsymbol{u}(\boldsymbol{v}_{j-1/2}^-),\boldsymbol{u}(\boldsymbol{v}_{j+1/2}^+)\Big)\cdot\boldsymbol{w}(x_{j-1/2}^+)\mathrm{d}t\Bigg)=0$$

for all $\boldsymbol{w}$ in V.
More compactly

$$B^{DG}(\boldsymbol{v},\boldsymbol{w})=0$$

Properties

This leads to entropy stability (use $\boldsymbol{w} = \boldsymbol{v}$, suitable boundary conditions):

$$\begin{aligned}\sum_{j\in J}\int_{T_j} S_-^N \mathrm{d}x = &\sum_{j\in J}\int_{T_j} S_-^0 \mathrm{d}x \\ &- \left\langle {}_{j+1/2,t}\frac{D(v^+ - v^-)}{2}, \frac{v^+ - v^-}{2}\right\rangle \\ &- \left\langle {}_{j-1/2,t}\frac{D(v^+ - v^-)}{2}, \frac{v^+ - v^-}{2}\right\rangle \\ &- \left\langle {}_{x,n}\tfrac{1}{2}S_{uu}(\theta^n)(u_- - u_+), u_- - u_+\right\rangle \\ \leq &\sum_{j\in J}\int_{T_j} S_-^0 \mathrm{d}x\end{aligned}$$

However, at discontinuities (shocks) this still leads to oscillations. That is why we introduce the streamline diffusion / shock capturing.

Streamline Diffusion ([JS87]; [JSH90])

Add the term

$$B^{SD}(\boldsymbol{v},\boldsymbol{w}) = \sum_{n=0}^{N-1}\sum_{j\in J}\int_{I^n}\int_{T_j}(\boldsymbol{u_v}\boldsymbol{w}_t + \boldsymbol{f}(\boldsymbol{u})_{\boldsymbol{v}}\boldsymbol{w}_x)\cdot\boldsymbol{D}^{SD}(\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x)\,\mathrm{d}x\mathrm{d}t$$

$$B^{DG}(\boldsymbol{v},\boldsymbol{w}) + B^{SD}(\boldsymbol{v},\boldsymbol{w}) = 0$$

with

$$\boldsymbol{D}^{SD} = \mathcal{O}(\Delta x)$$

and $\boldsymbol{D}^{SD}$ positive semidefinite.
Leads to additional diffusion proportional to

$$(\boldsymbol{u_v}\boldsymbol{v}_t + \boldsymbol{f}(\boldsymbol{u})_{\boldsymbol{v}}\boldsymbol{v}_x)\cdot\boldsymbol{D}(\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x) = (\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x)\cdot\boldsymbol{D}^{SD}(\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x)$$

Shock-capturing (Barth)

Idea: at shocks the residual is big (is it?)
add (homogeneous) diffusion proportional to the residual

$$B^{SC}(\boldsymbol{v},\boldsymbol{w}) = \Delta x \sum_{n=0}^{N-1} \sum_{j\in J} \int_{I^n} \int_{T_j} \epsilon_j^n (\boldsymbol{u}_t \cdot \boldsymbol{w}_t + \boldsymbol{u}_x \cdot \boldsymbol{w}_x)\,\mathrm{d}x\mathrm{d}t$$

$$B^{DG}(\boldsymbol{v},\boldsymbol{w}) + B^{SD}(\boldsymbol{v},\boldsymbol{w}) + B^{SC}(\boldsymbol{v},\boldsymbol{w}) = 0$$

$$\epsilon_j^n = \frac{\Delta x^{1-\alpha}\left(C_i R_{i,j}^n + \Delta x^{-1/2-\beta} C_b R_{b,j}^n\right)}{\sqrt{\int_{I^n}\int_{T_j}(\boldsymbol{v}_t \cdot \boldsymbol{u}_{\boldsymbol{v}} \boldsymbol{v}_t + \boldsymbol{v}_x \cdot \boldsymbol{u}_{\boldsymbol{v}} \boldsymbol{v}_x)\,\mathrm{d}x\mathrm{d}t} + \Delta x}$$

$$R_{i,j}^n = \sqrt{\int_{I^n}\int_{T_j} (\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x) \cdot \boldsymbol{u}_{\boldsymbol{v}}^{-1} (\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x)\mathrm{d}x\mathrm{d}t}$$

parameter values: $C_i = 1,\ \alpha = 0,\ C_b = 0$

Properties - Goals revisited

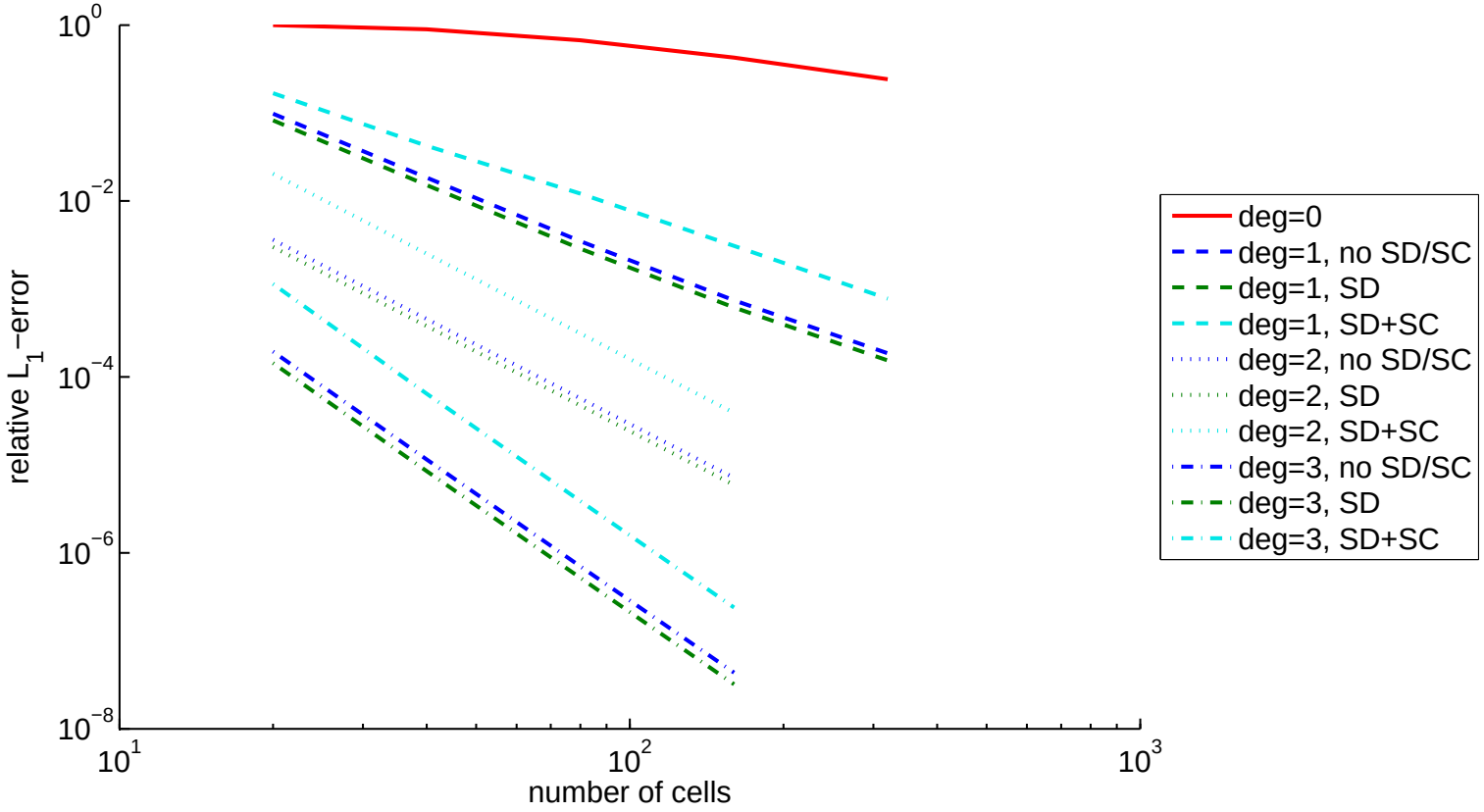
- formally arbitrarily high-order accurate
- entropy stable (without/with SD or SC)
- (global) maximum principle for the scalar case (using a logarithmic entropy)
- convergence for the scalar case?
- multidimensional
- reasonably efficient?

Implementation

- currently in MATLAB
- quite slow
- For each time interval we have to solve a non-linear system for the dofs associated to it.
- currently: mostly by a damped Newton method ($\Rightarrow$ we have to compute the Jacobian)
- planned: Newton-Krylov method ($\Rightarrow$ we have to compute only the multiplication with the Jacobian)
  preconditioner?

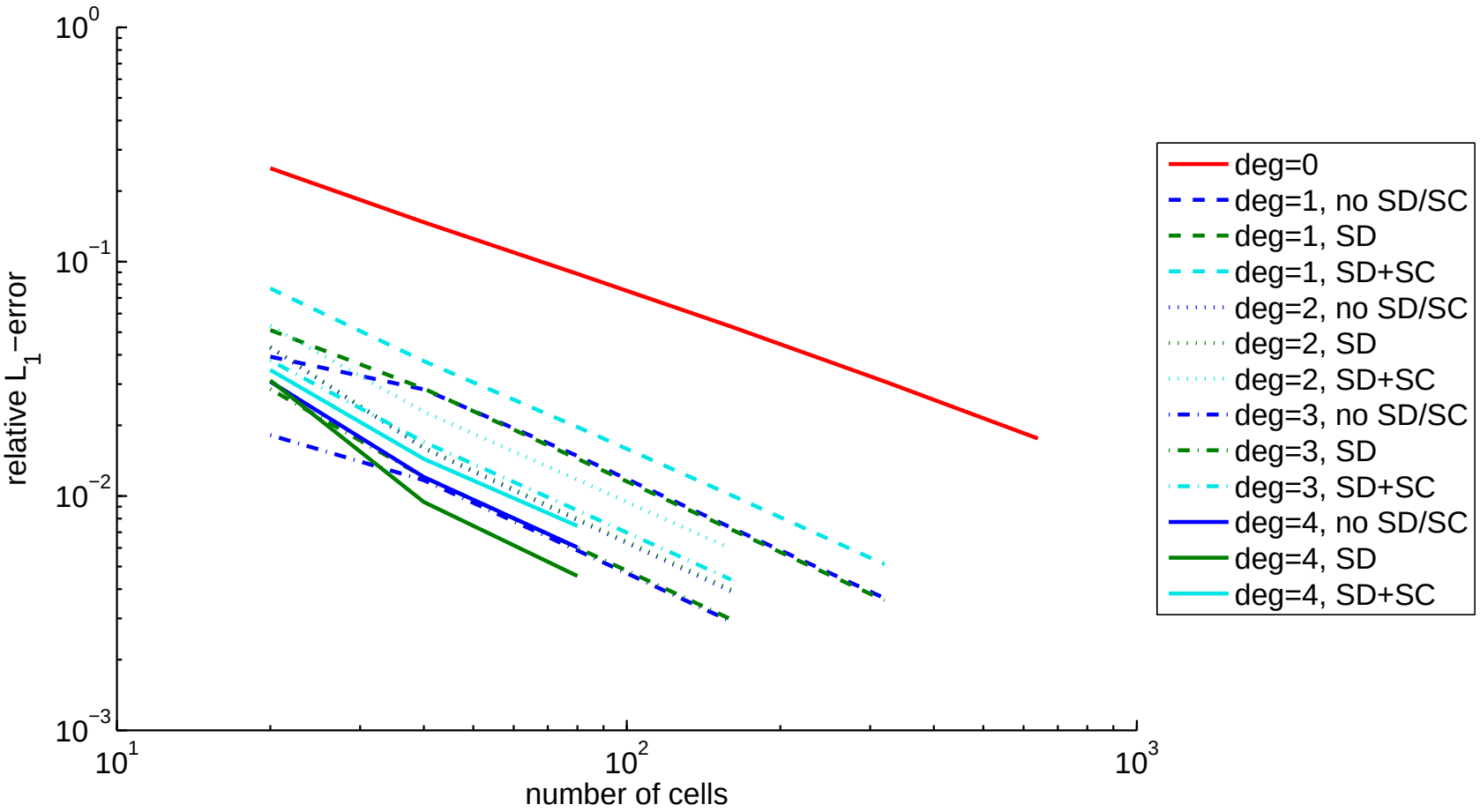
Wave equation (smooth initial data)

$$h_t + cm_x = 0$$
$$m_t + ch_x = 0$$



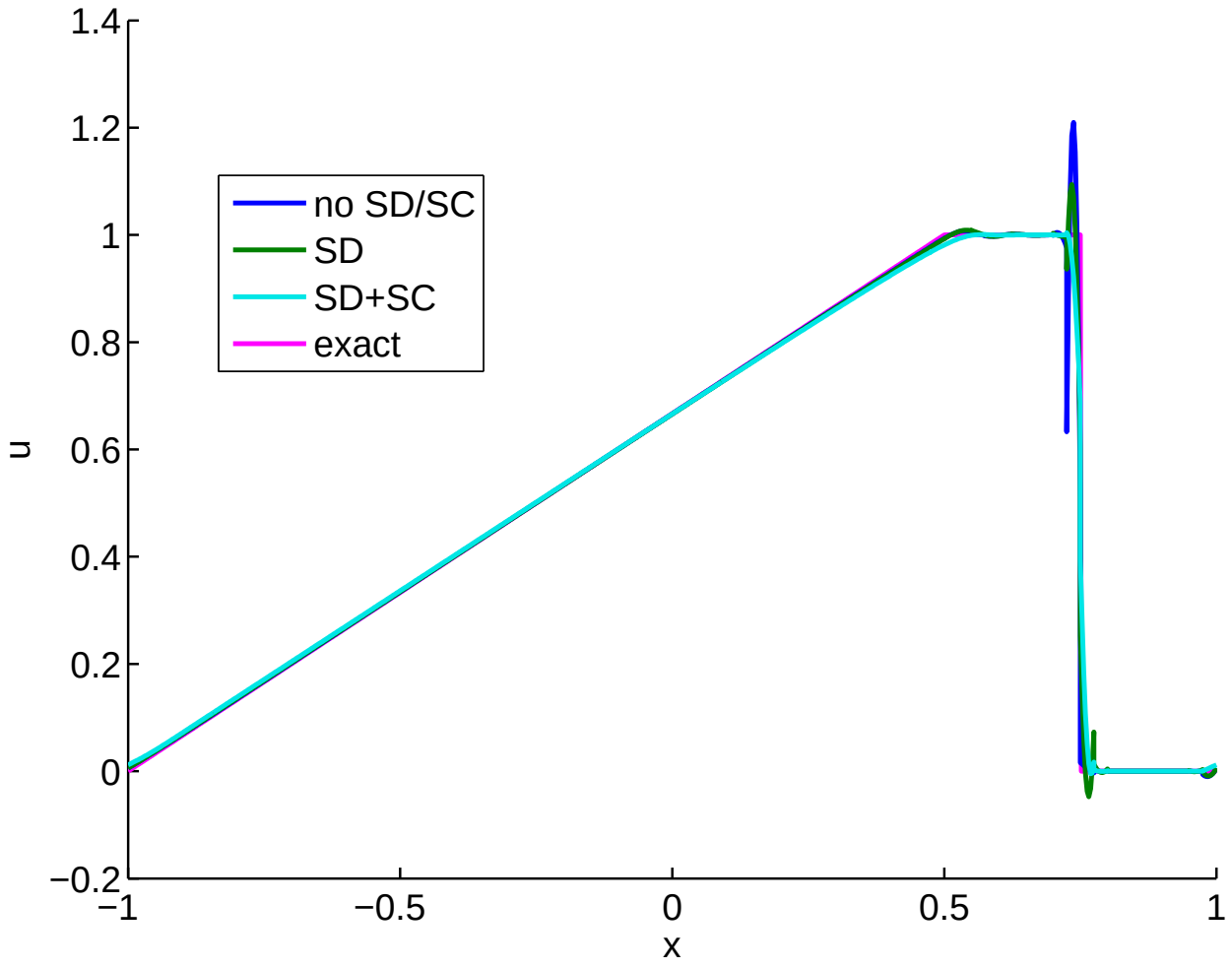
Burgers' equation

$$u_t + (u^2/2)_x = 0$$



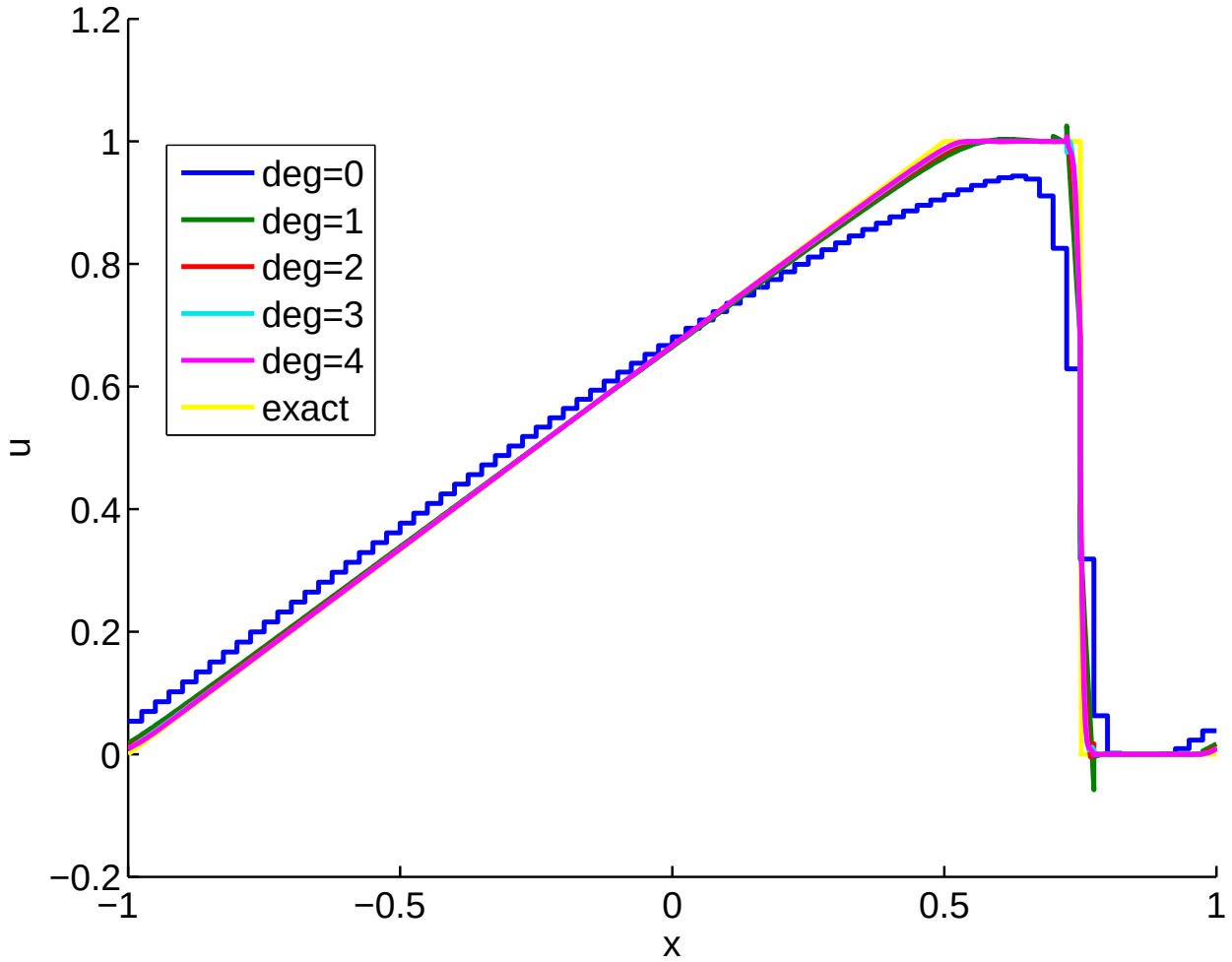
Burgers' equation

$deg = 2,\ N_x = 80$

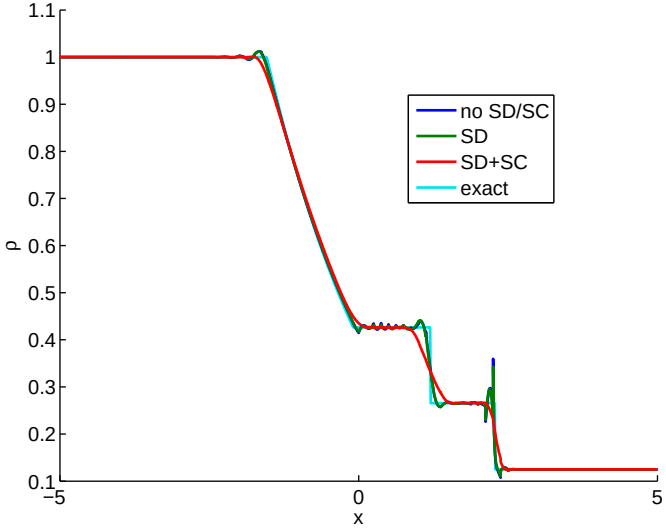


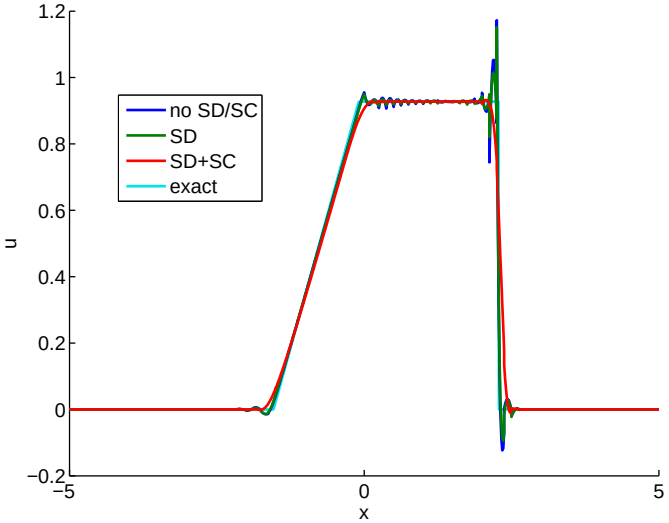
Burgers' equation

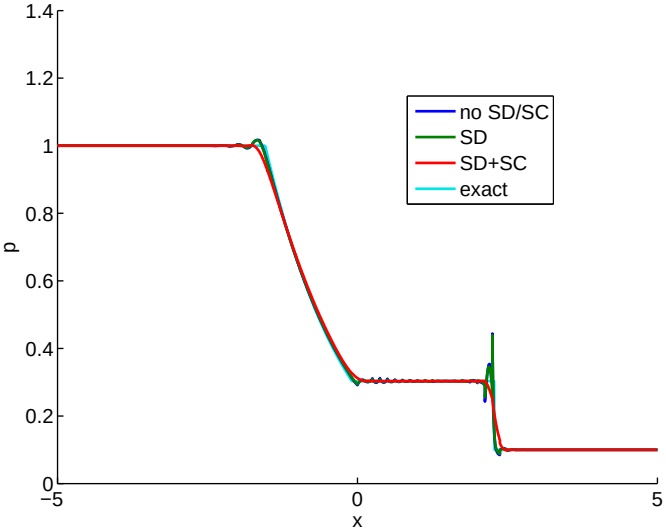
$SD + SC,\ N_x = 80$



Euler equations - Sod shock tube



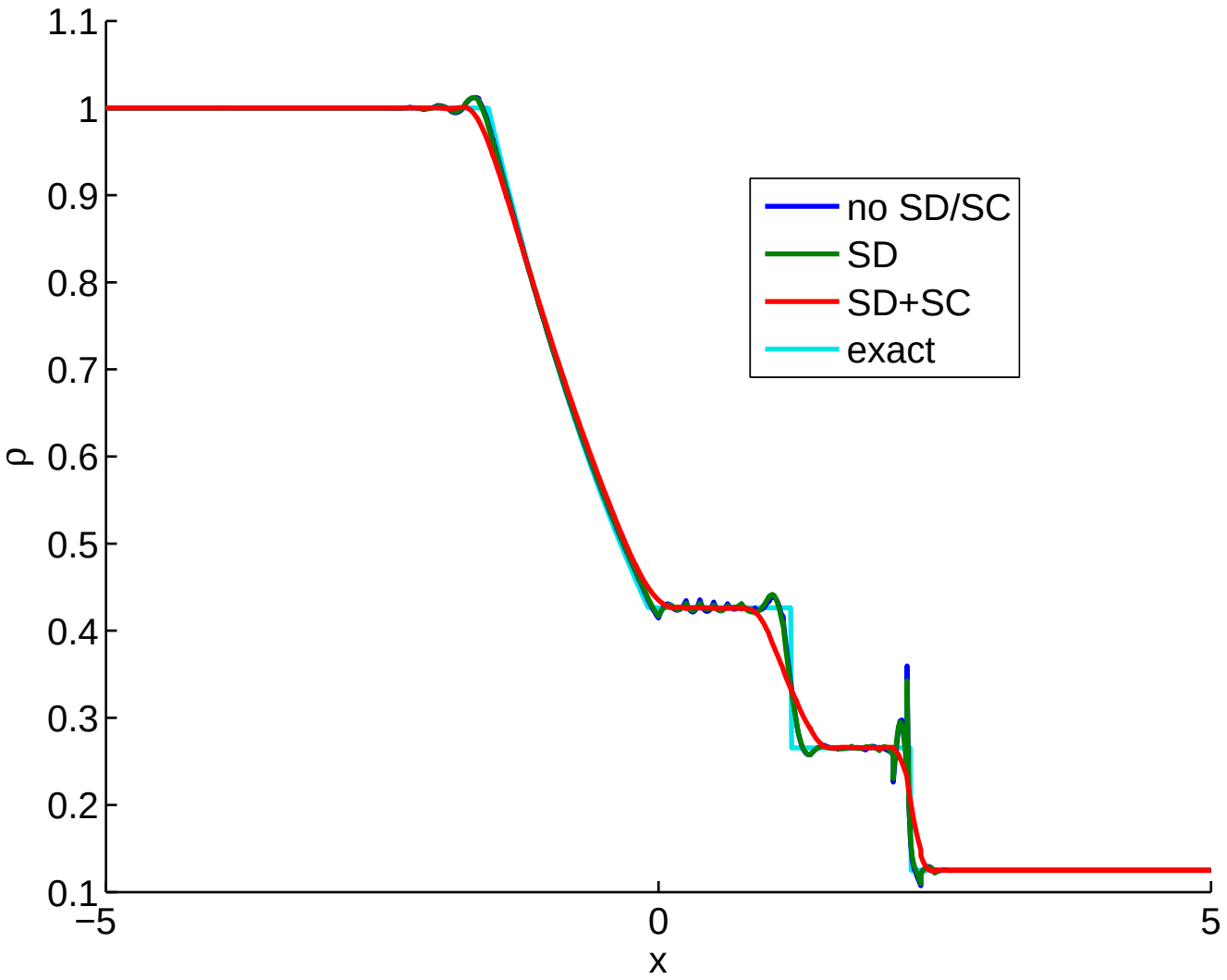




$N_x = 80,\ deg = 2$

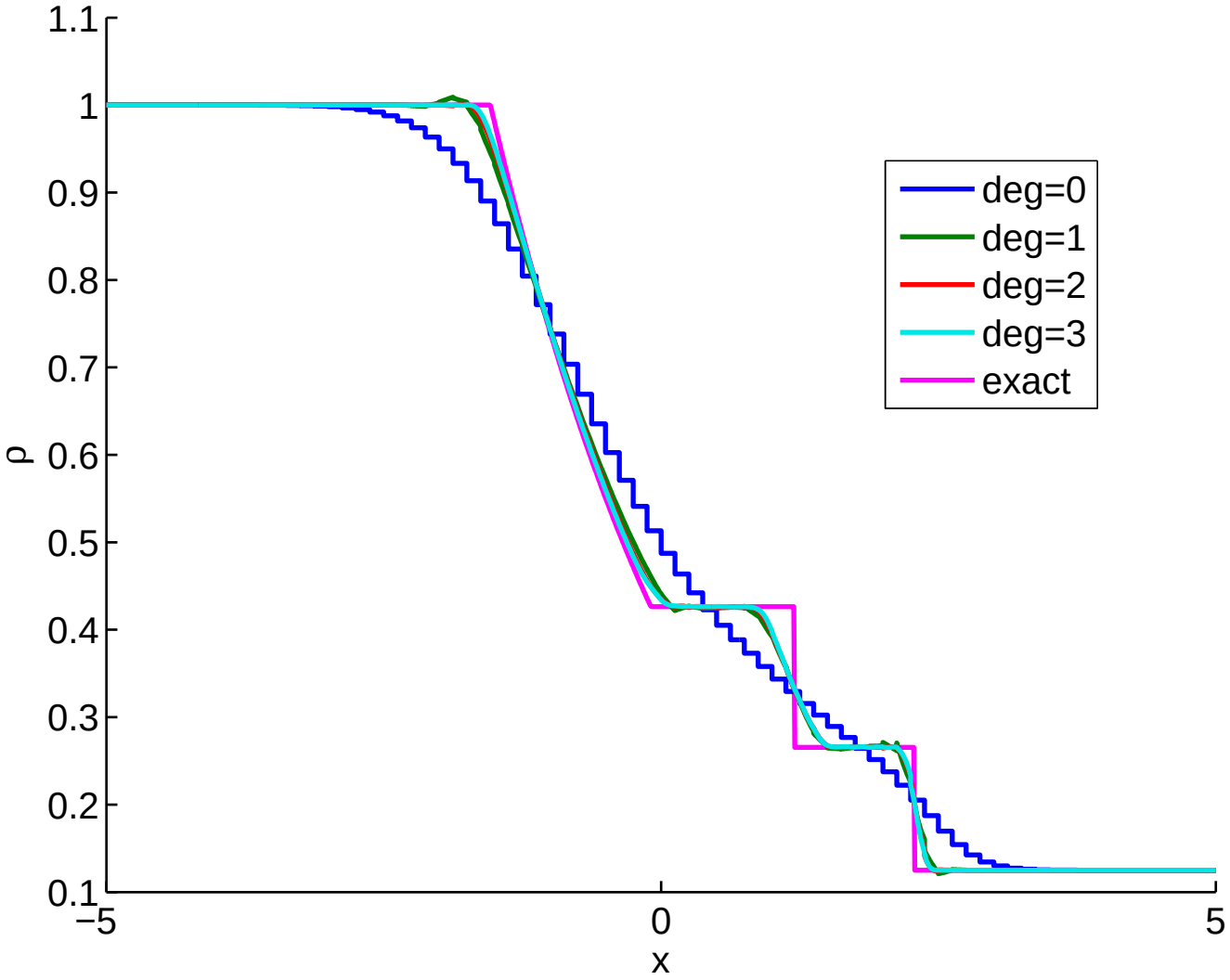
Euler equations - Sod shock tube

$N_x = 80,\ deg = 2$

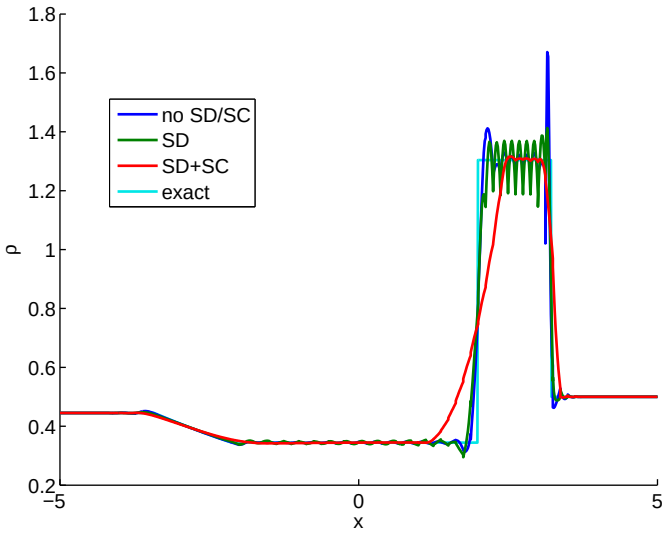


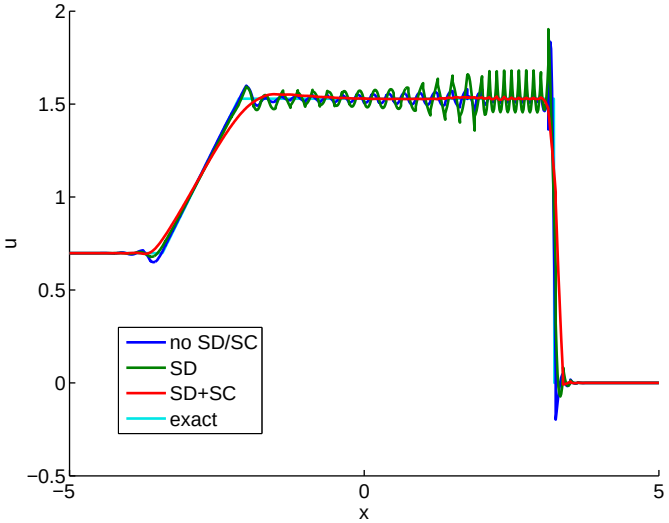
Euler equations - Sod shock tube

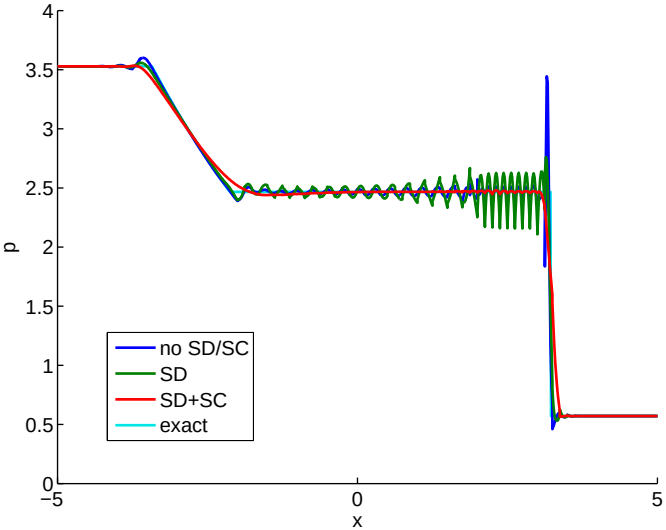
$N_x = 80$, SD+SC



Euler equations - Lax shock tube



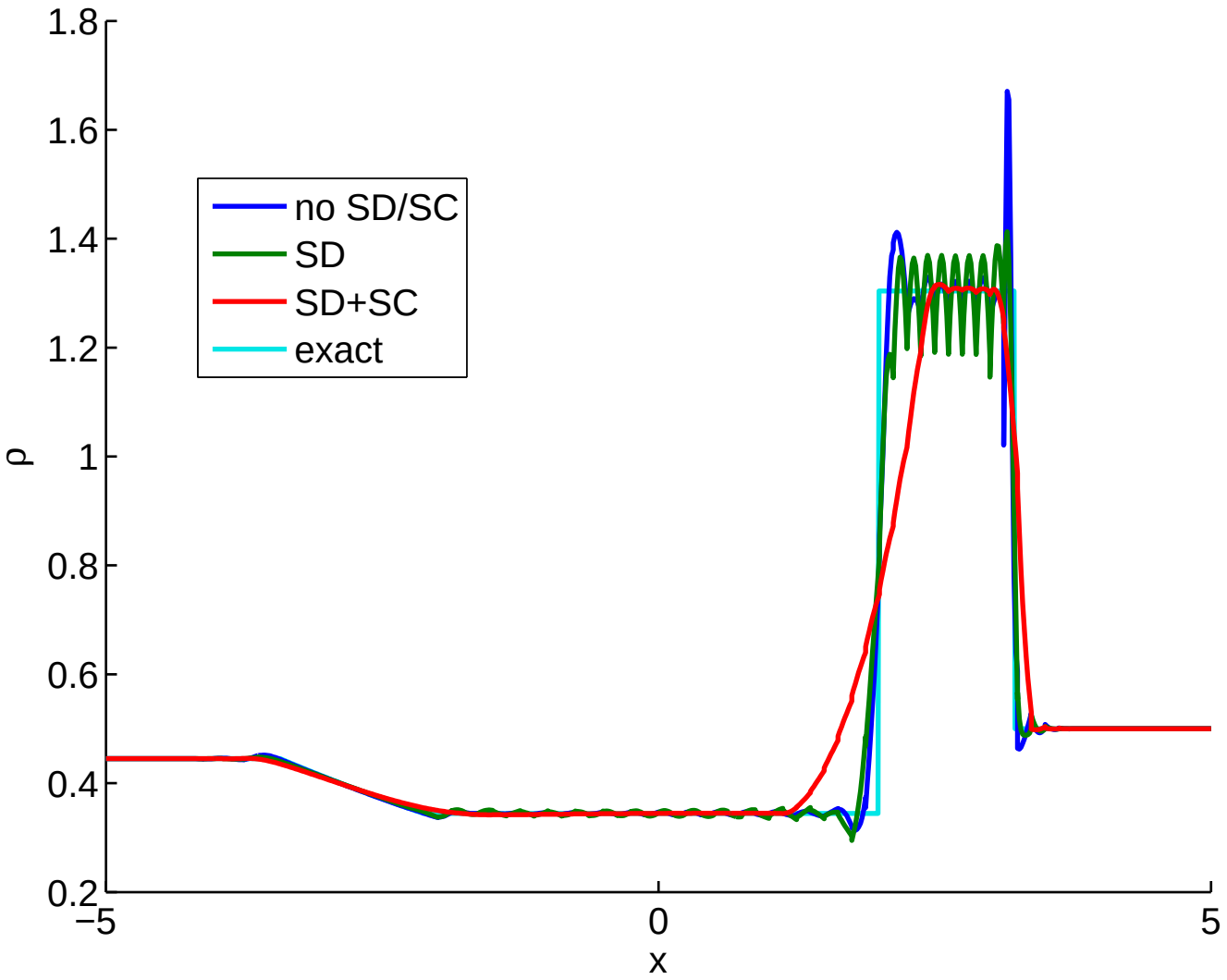




$N_x = 80,\ deg = 2$

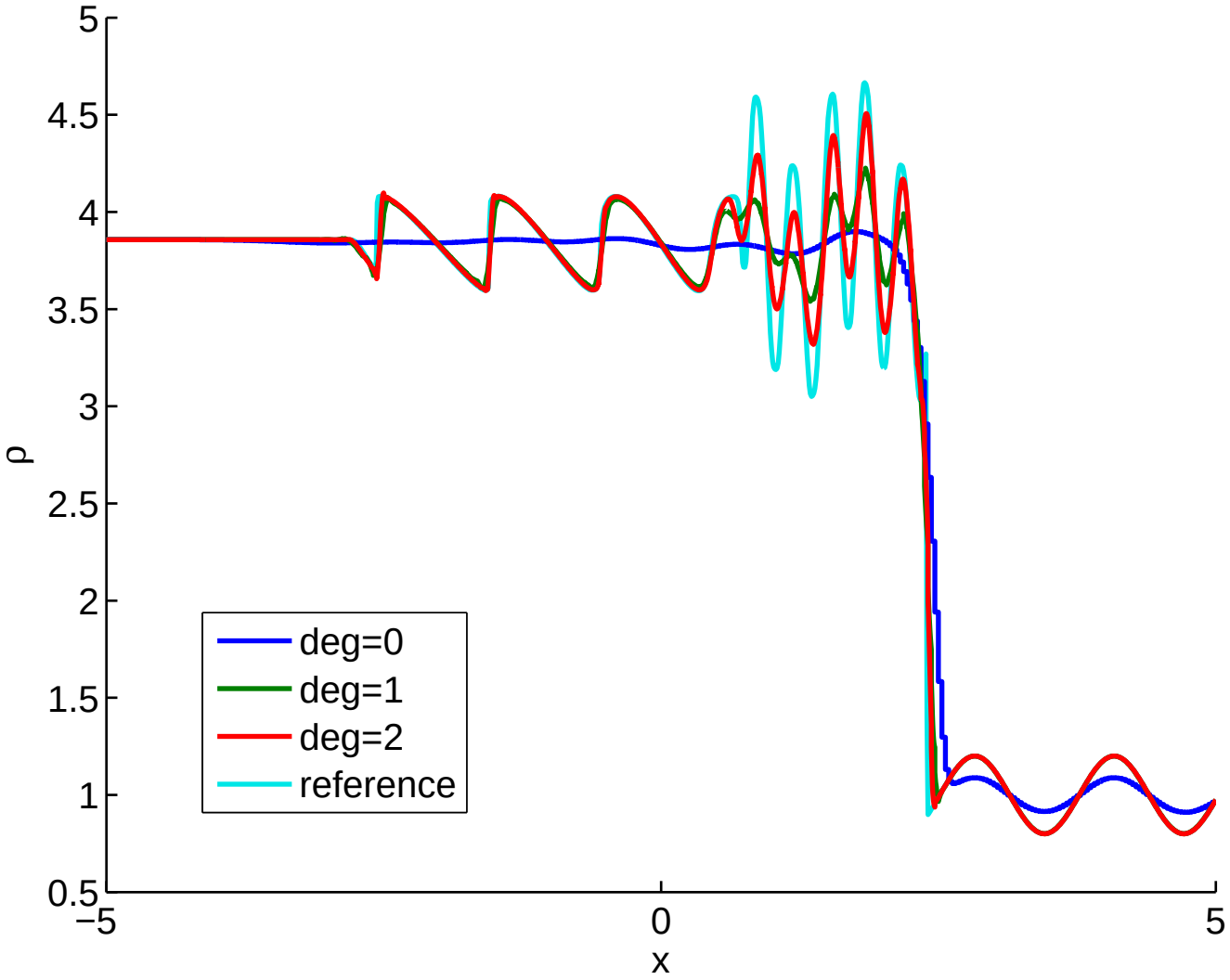
Euler equations - Lax shock tube

$N_x = 80,\ deg = 2$



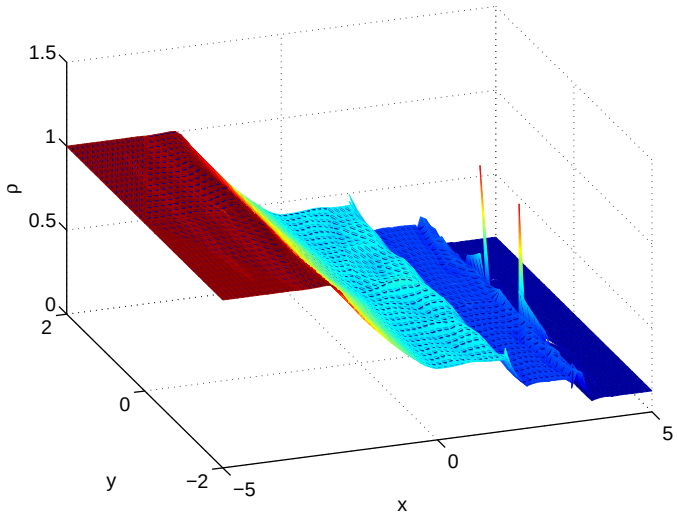
Euler equations - shock-entropy wave interaction

$N_x = 320$, SD+SC

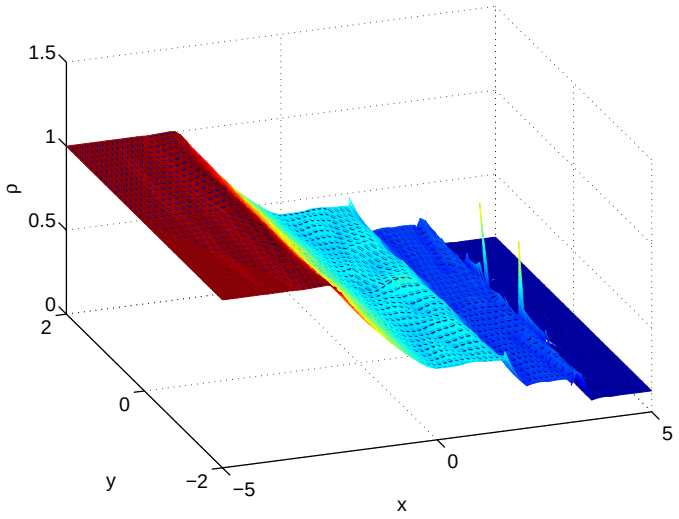


Euler equations in 2D - Sod shock tube

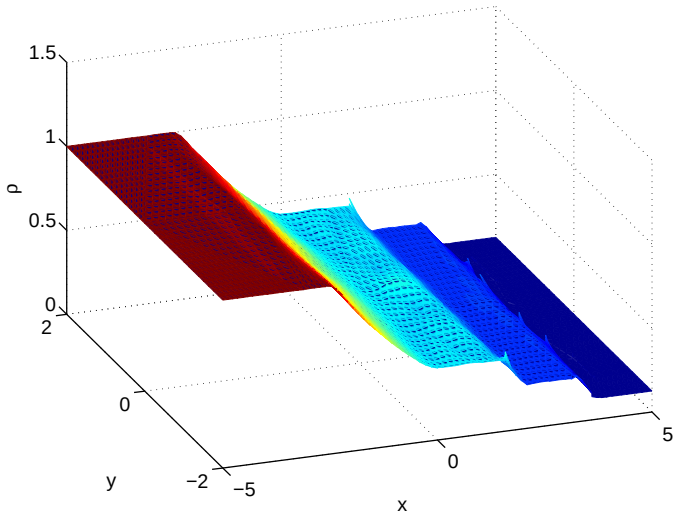
no SD/SC



SD



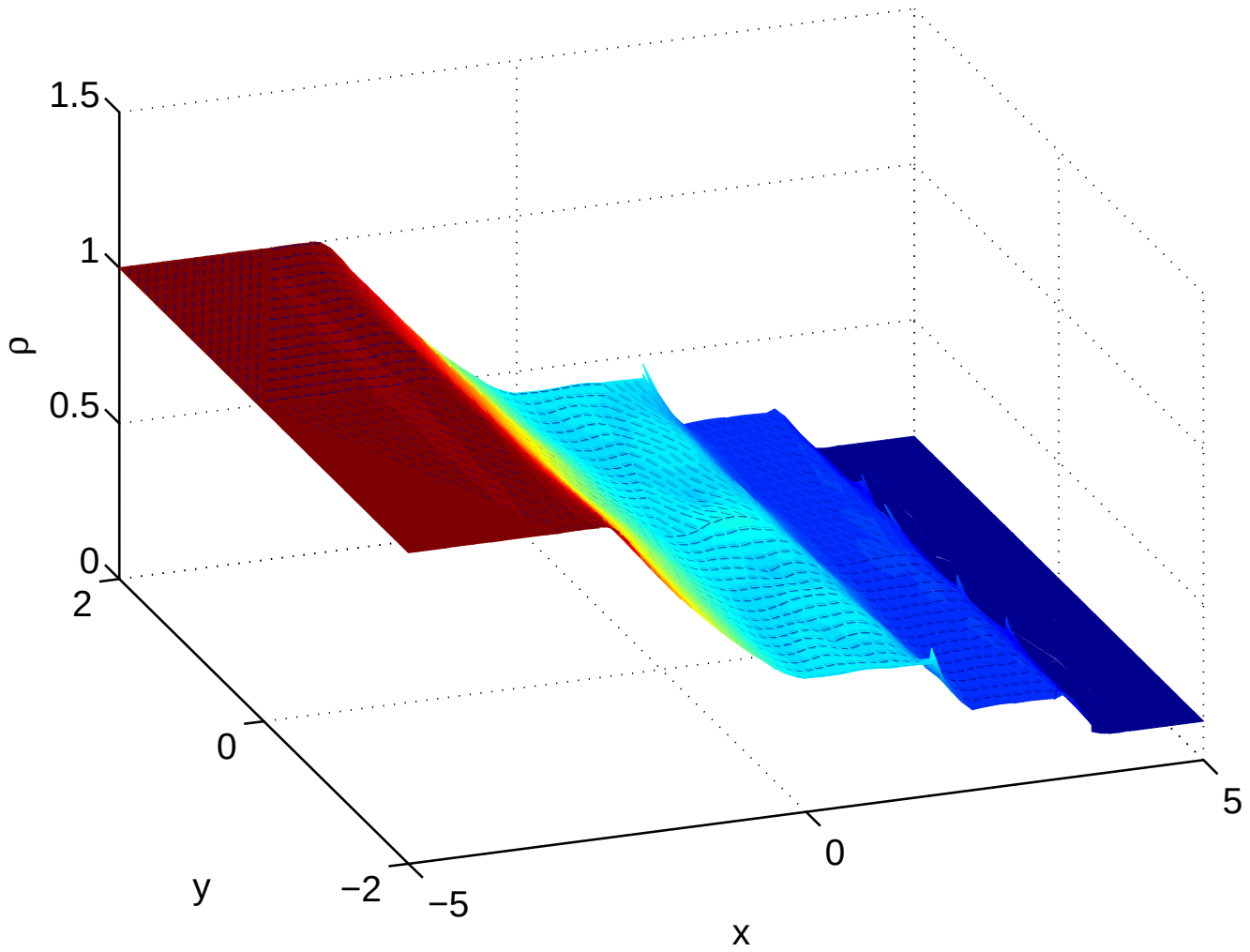
SD+SC



$N_c = 288,\ deg = 2$

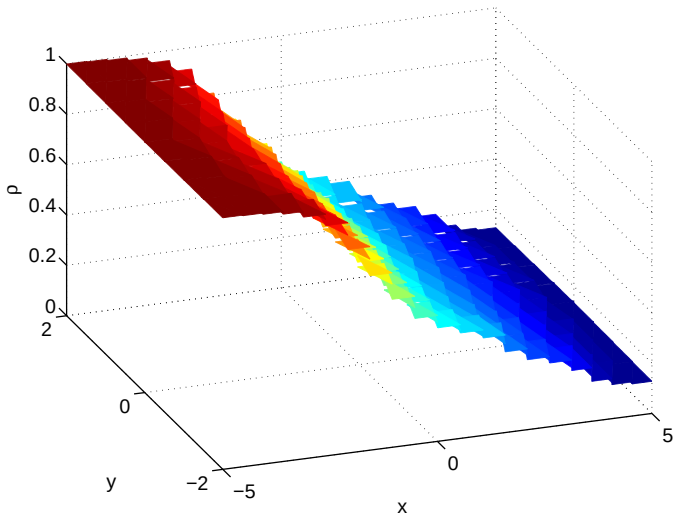
Euler equations in 2D - Sod shock tube

SD+SC, $N_c = 288$, $deg = 2$

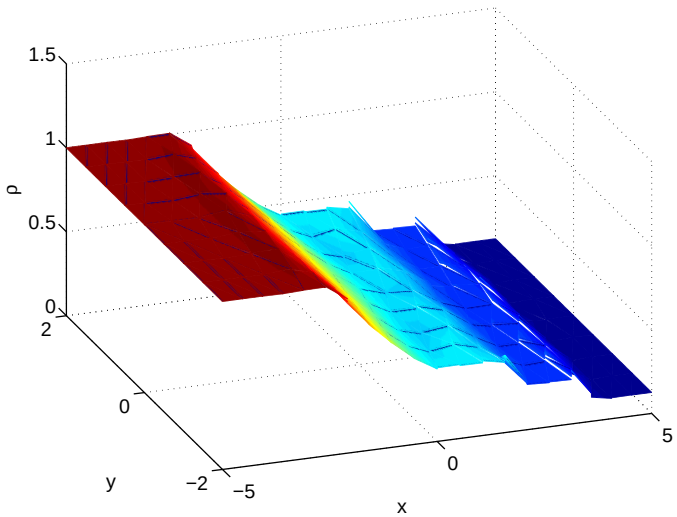


Euler equations in 2D - Sod shock tube

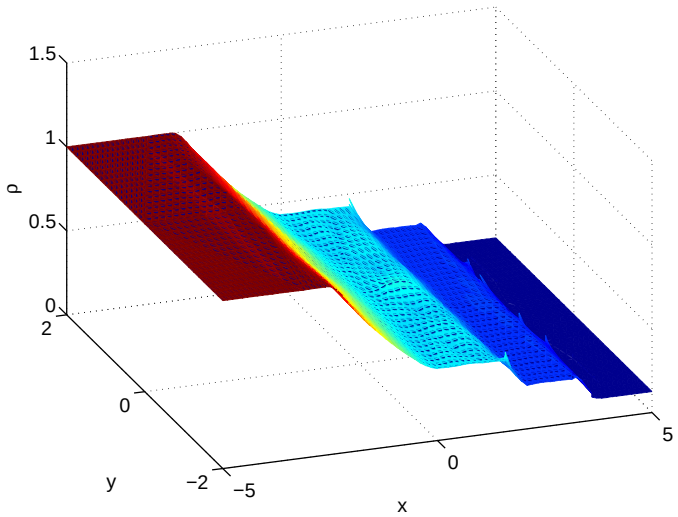
$deg = 0$



$deg = 1$



$deg = 2$



$N_c = 288$, SD+SC

Conclusions

- we get an entropy-stable DG FE method by:
  discretizing entropy variables
  using entropy stable numerical fluxes
- the solution is quite oscillatory at discontinuities
- by streamline diffusion and shock-capturing we get a much less oscillatory solution,
  but it is quite diffusive at contact discontinuities
- parameters and exact form of shock-capturing?
- convergence proofs?
- implementation: efficient solution of the non-linear systems?

Bibliography

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The numerical viscosity of entropy stable schemes for systems of conservation laws. i.
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Appendix

SC - boundary residual

$$R_{b,j}^n =$$

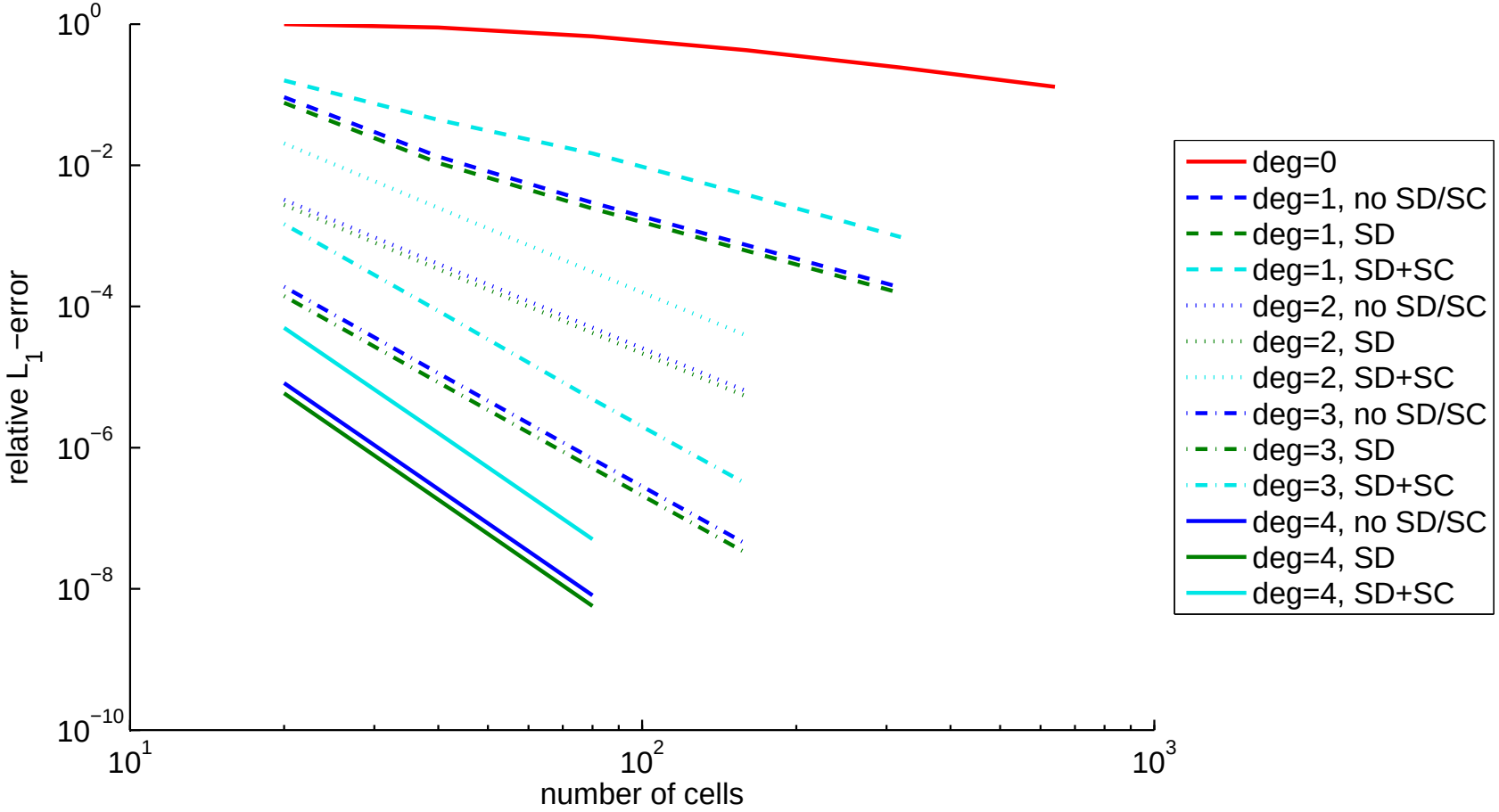
$$\Bigg(\int_{T_j} \left(\boldsymbol{u}_+^n - \boldsymbol{u}_-^n\right) \cdot \boldsymbol{u}_{\boldsymbol{v}}^{-1} \left(\boldsymbol{u}_+^n - \boldsymbol{u}_-^n\right) \mathrm{d}x$$

$$+ \int_{I^n} \left(\boldsymbol{F}(\boldsymbol{u}_{j+1/2}^-, \boldsymbol{u}_{j+1/2}^+) - \boldsymbol{f}(\boldsymbol{u}_{j+1/2}^-) \right) \cdot \boldsymbol{u}_{\boldsymbol{v}}^{-1} \left(\boldsymbol{F}(\boldsymbol{u}_{j+1/2}^-, \boldsymbol{u}_{j+1/2}^+) - \boldsymbol{f}(\boldsymbol{u}_{j+1/2}^-) \right) \mathrm{d}t$$

$$+ \int_{I^n} \left(\boldsymbol{F}(\boldsymbol{u}_{j-1/2}^-, \boldsymbol{u}_{j-1/2}^+) - \boldsymbol{f}(\boldsymbol{u}_{j-1/2}^+) \right) \cdot \boldsymbol{u}_{\boldsymbol{v}}^{-1} \left(\boldsymbol{F}(\boldsymbol{u}_{j-1/2}^-, \boldsymbol{u}_{j-1/2}^+) - \boldsymbol{f}(\boldsymbol{u}_{j-1/2}^+) \right) \mathrm{d}t \Bigg)^{\frac{1}{2}}$$

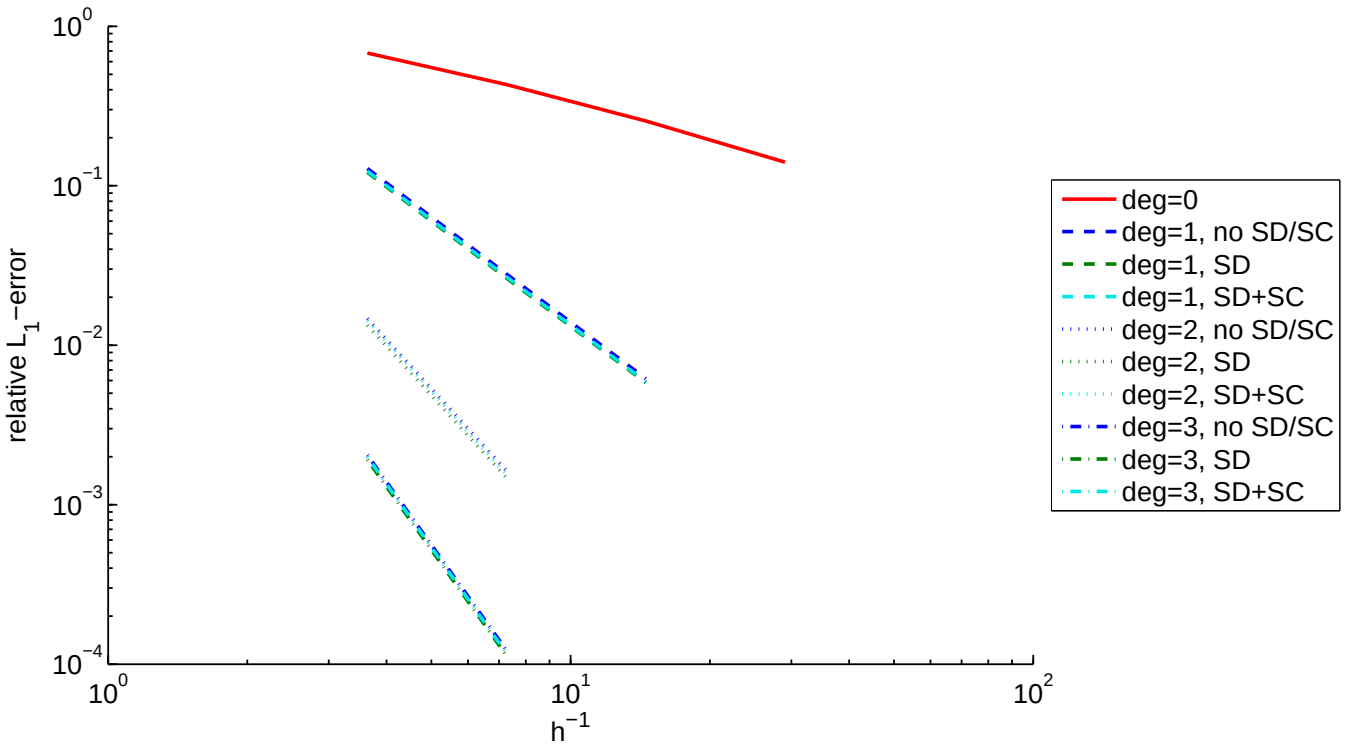
Linear advection

$$u_t + a u_x = 0$$



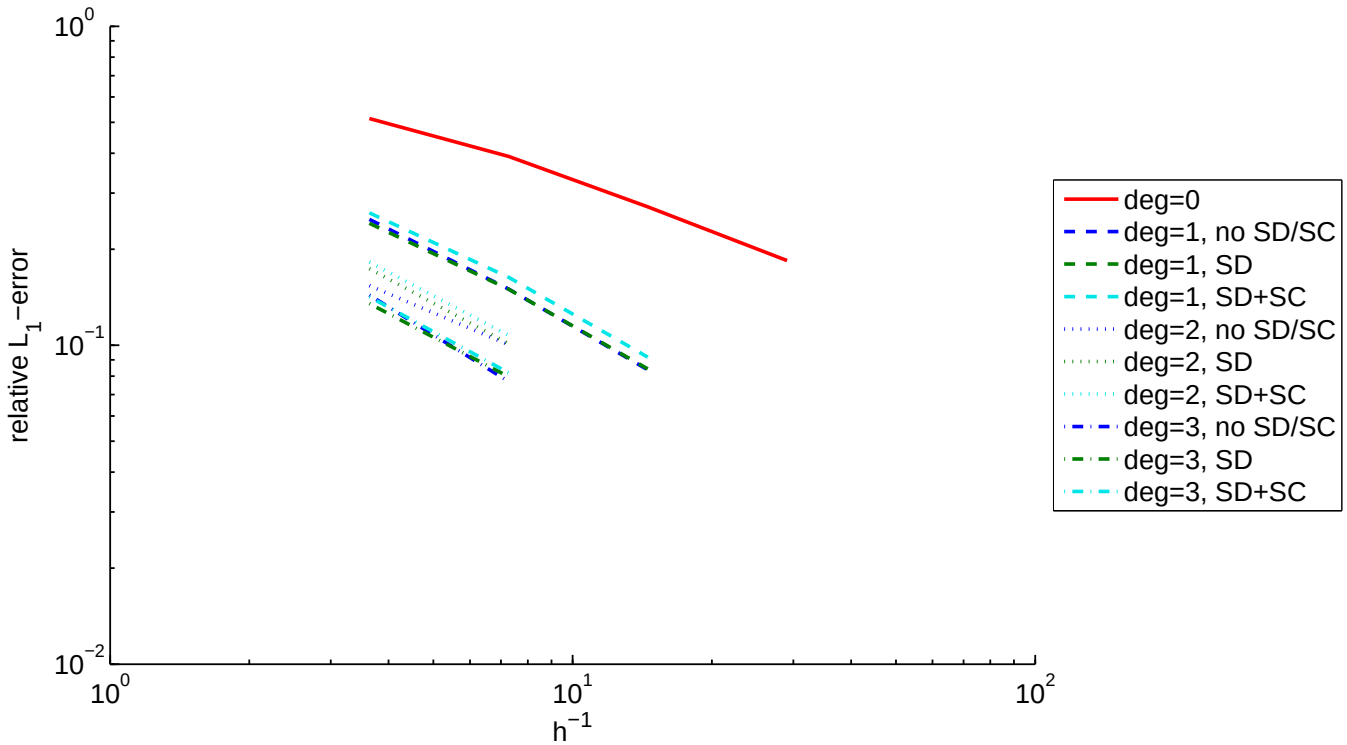
Linear advection equation in 2D

$$u_t + au_x + bu_y = 0$$



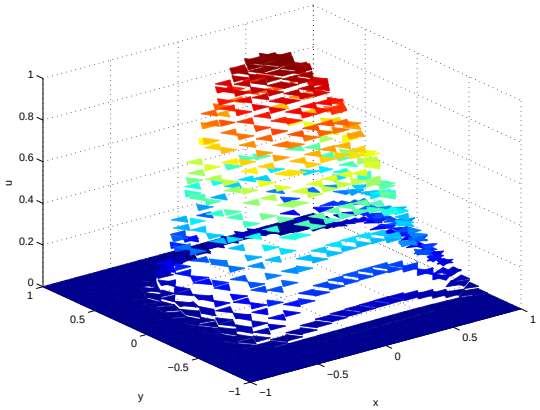
Burgers'/Advection equation in 2D

$$u_t + au_x + (u^2/2)_y = 0$$

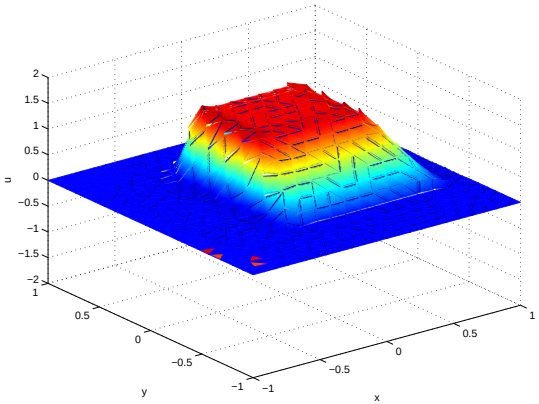


Burgers'/Advection equation in 2D

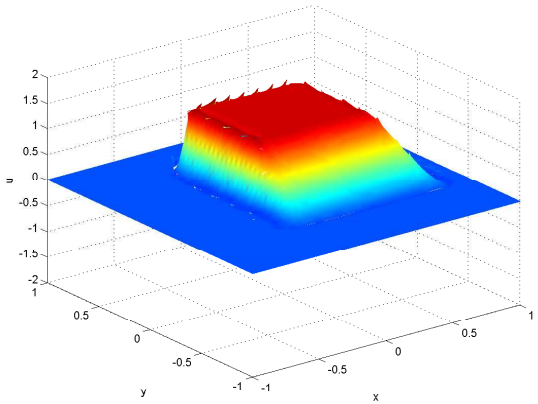
$deg = 0$



$deg = 1$



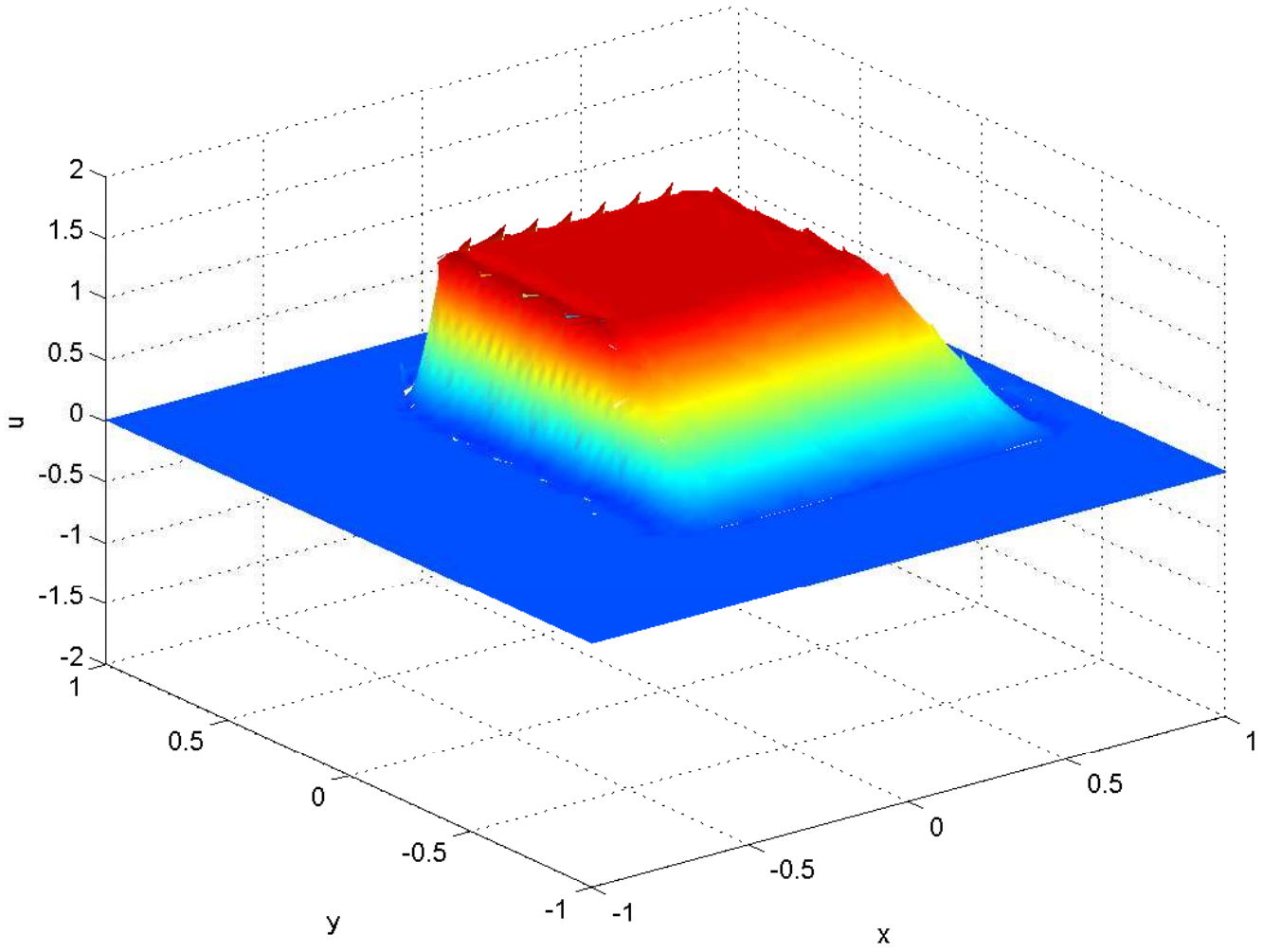
$deg = 2$



$N_c = 840$, SD+SC

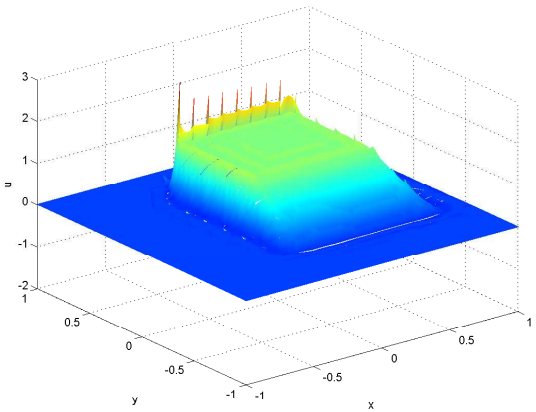
Burgers'/Advection equation in 2D

$deg = 2,\ N_c = 840,$ SD+SC

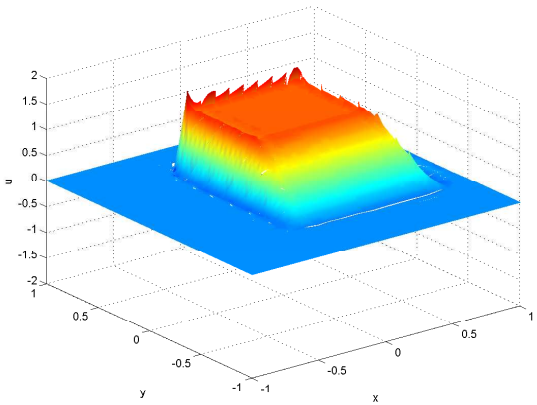


Burgers'/Advection equation in 2D

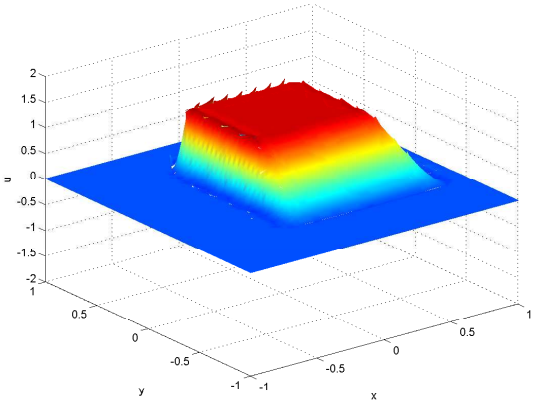
no SD/SC



SD



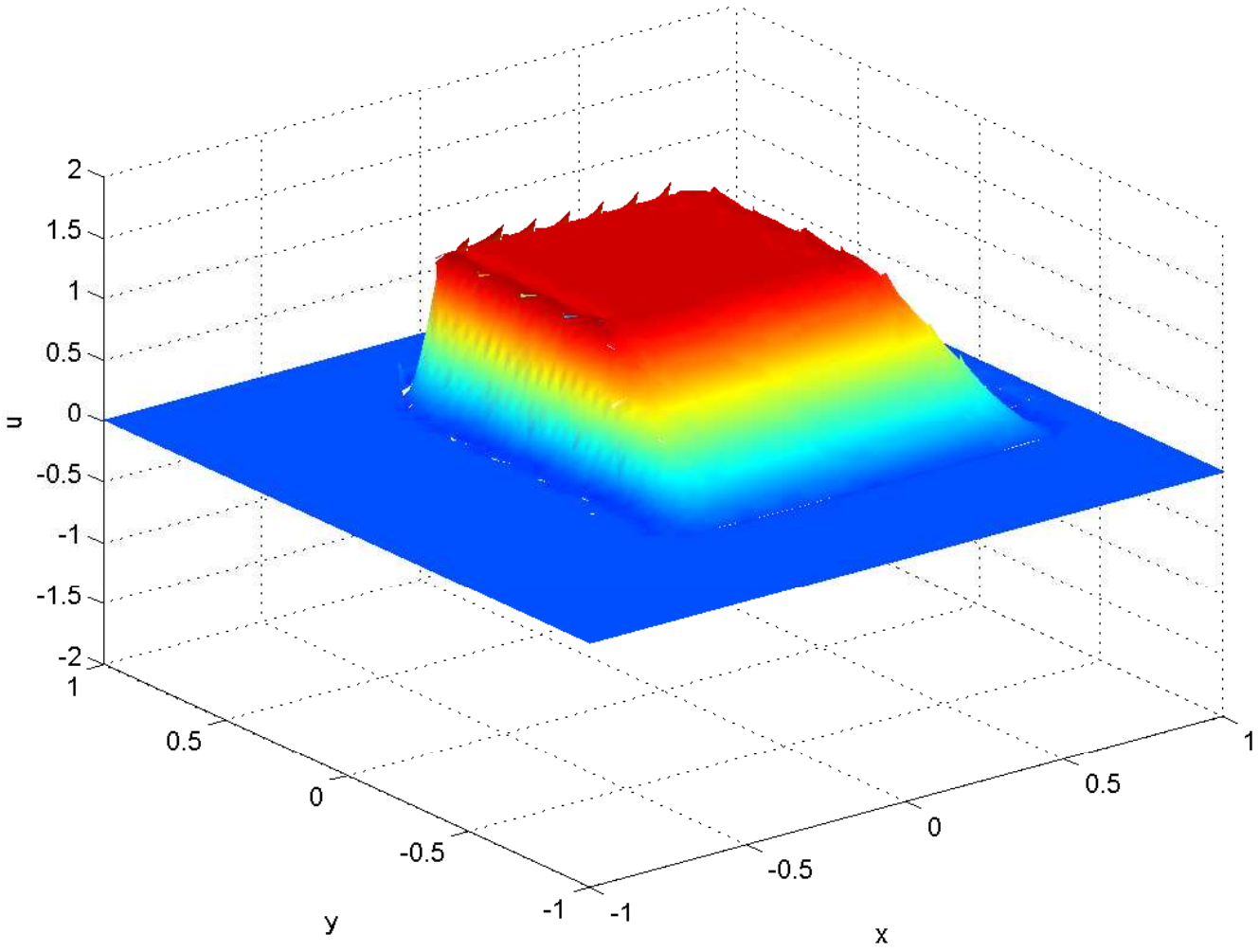
SD+SC



$N_c = 840,\ deg = 2$

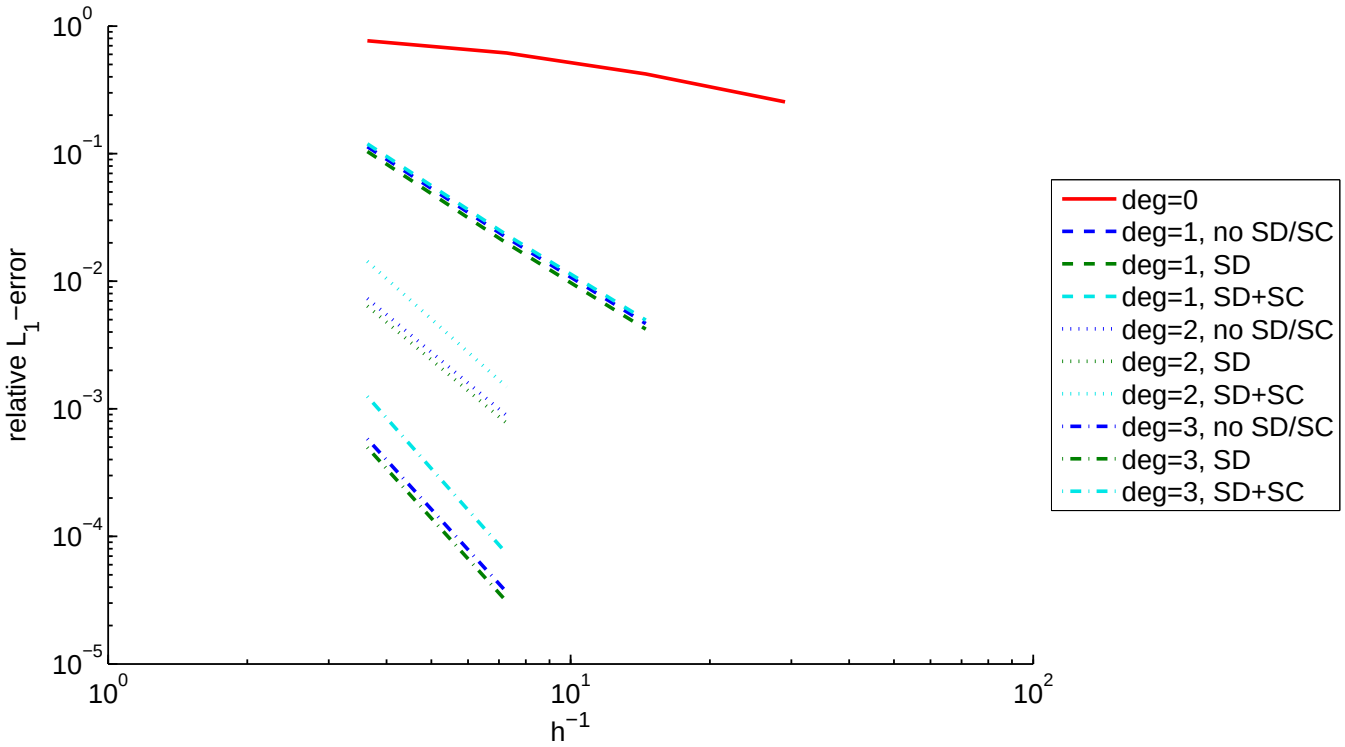
Burgers'/Advection equation in 2D

SD+SC, $N_c = 840$, $deg = 2$

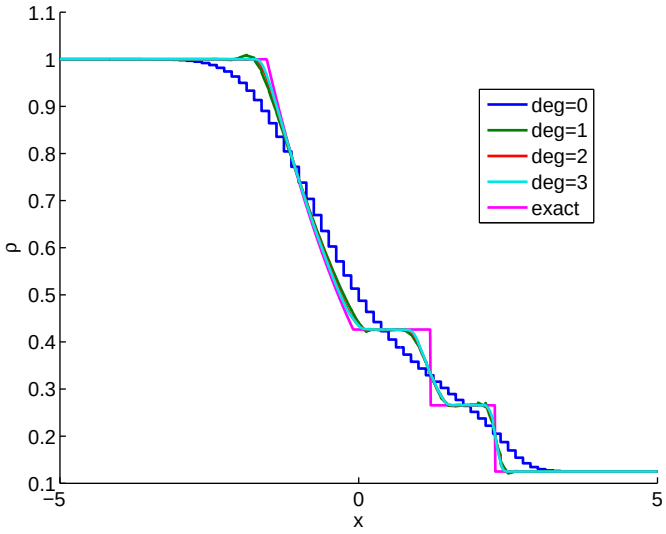


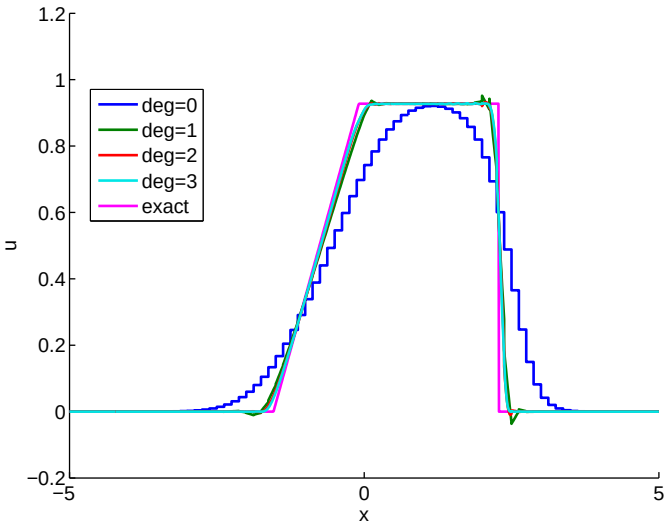
Wave equation in 2D

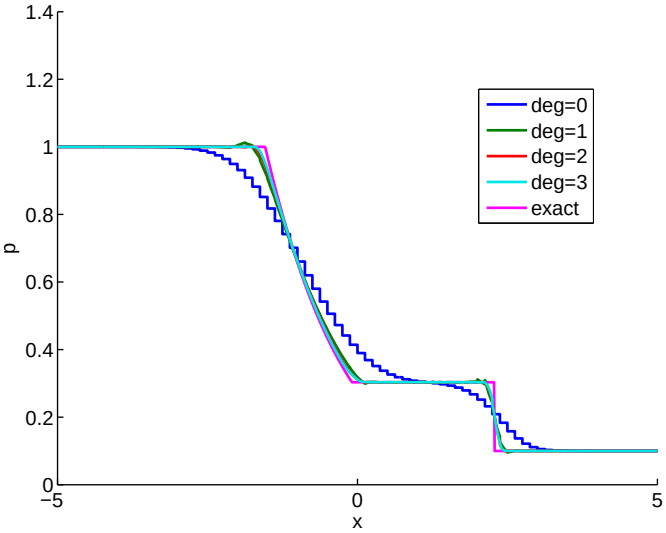
$$
\begin{aligned}
h_t + cm_x + cn_y &= 0 \\
m_t + ch_x &= 0 \\
n_t + ch_y &= 0
\end{aligned}
$$



Euler equations - Sod shock tube



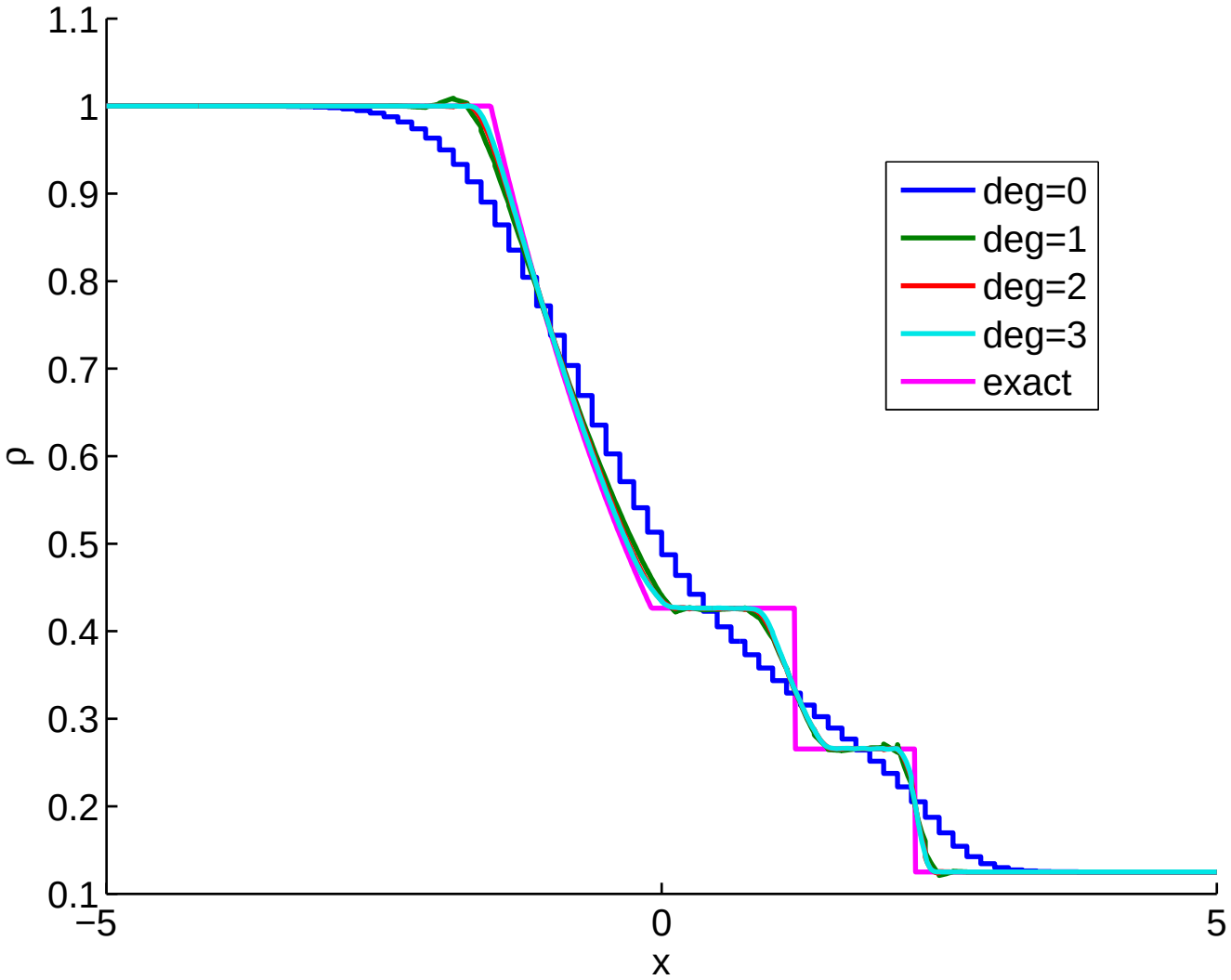




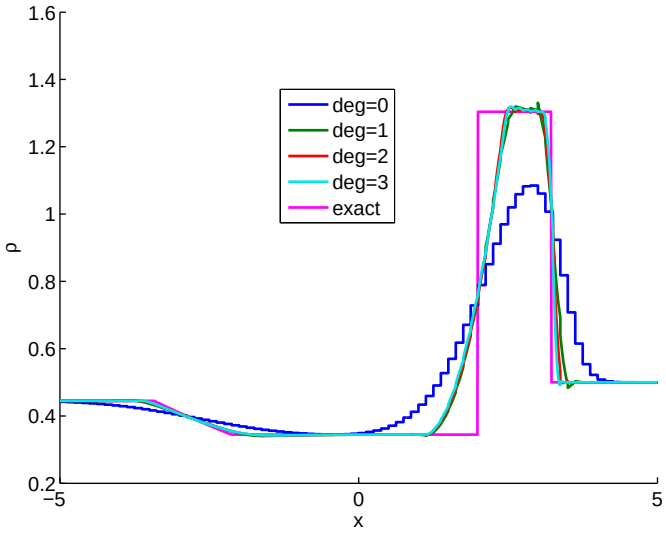
$N_x = 80$, SD+SC

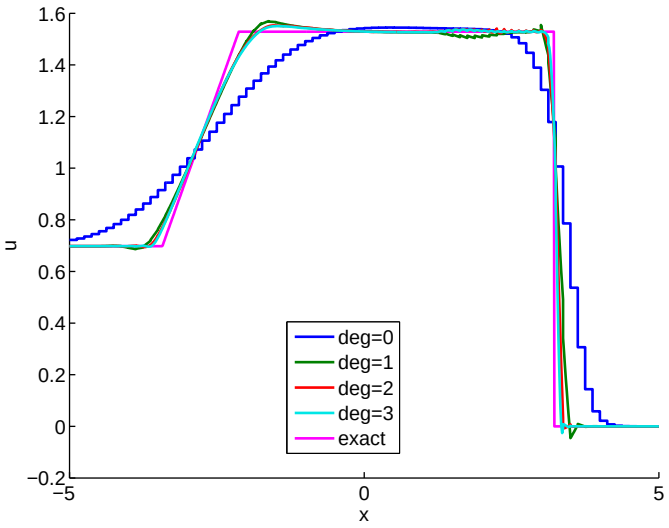
Euler equations - Sod shock tube

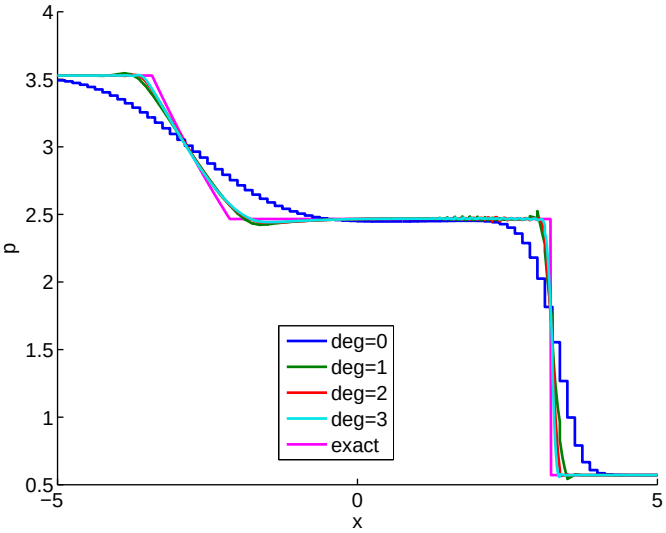
$N_x = 80$, SD+SC



Euler equations - Lax shock tube



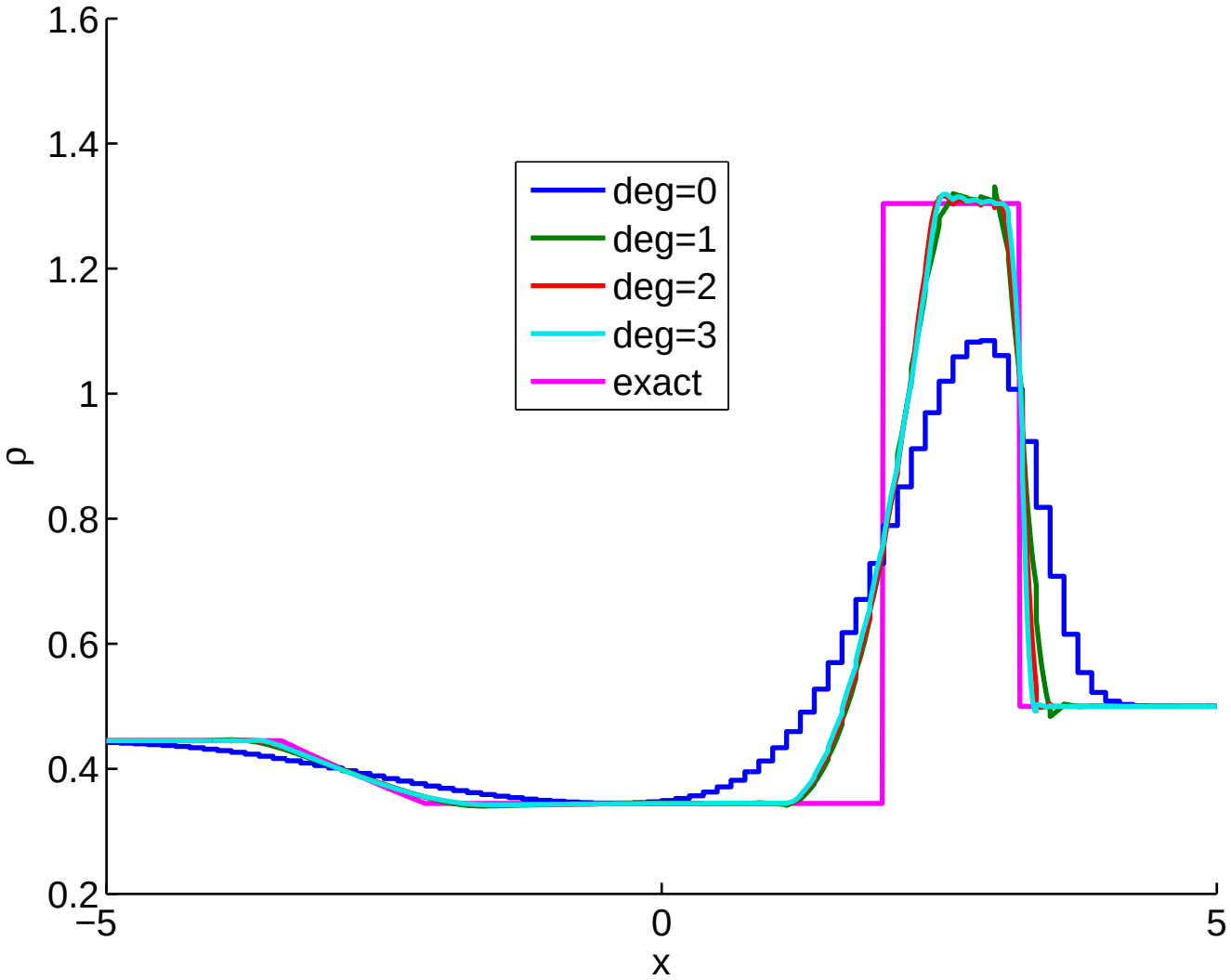




$N_x = 80$, SD+SC

Euler equations - Lax shock tube

$N_x = 80$, SD+SC



Euler equations in 2D - Sod shock tube