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SAM

Stable Numerical Scheme for the Magnetic Induction Equation with Hall Effect

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joint work with Siddhartha Mishra

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Outline

- Formulation and motivation of the problem
- Theoretical analysis
- Numerical methods
- Time integration
- DG Formulation

Magnetic Reconnection

- Change in topology of the magnetic field

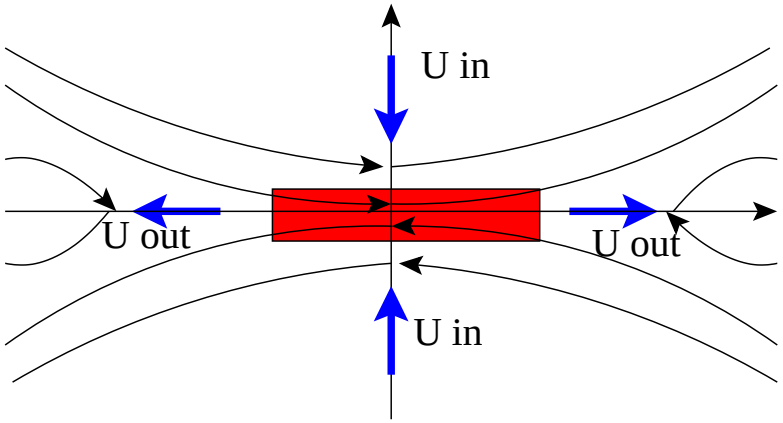

Figure: Schematic of a reconnection.

- Magnetic energy $\Rightarrow$ kinetic and thermal energy
- Dissipation

MHD Equations

The equations

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\nabla \cdot (\rho \mathbf{u}) \\
\frac{\partial (\rho \mathbf{u})}{\partial t} &= -\nabla \left\{ \rho \mathbf{u}\mathbf{u}^\top + \left(p + \frac{\mathbf{B}^2}{2} \right) \mathbf{I}_{3\times 3} - \mathbf{B}\mathbf{B}^\top \right\} \\
\frac{\partial \mathcal{E}}{\partial t} &= -\nabla \left\{ \left(\mathcal{E} + p - \frac{B^2}{2} \right) \mathbf{u} + \mathbf{E} \times \mathbf{B} \right\} \\
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E}
\end{aligned}
$$

are coupled through the equation of state

$$
\mathcal{E} = \frac{p}{\gamma - 1} + \frac{\rho \mathbf{u}^2}{2} + \frac{B^2}{2}
$$

To complete the formulation of the problem we need to state some equation for **E**

Ideal MHD

Standard model for E:
Ohm's Law

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B}$$

Problem:
no dissipation $\Rightarrow$ "frozen" condition.

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Resistive MHD:

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} - \eta \mathbf{J}$$

not sufficient for fast reconnection.

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not sufficient for fast reconnection.

We need another model...
Numerical simulation and laboratory experiment $\Rightarrow$ Hall Effect

Generalized Ohm's Law

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \eta \mathbf{J} + \frac{\delta_i}{L_0} \frac{\mathbf{J} \times \mathbf{B}}{\rho} - \frac{\delta_i}{L_0} \frac{\nabla \overleftrightarrow{p}}{\rho} + \left(\frac{\delta_e}{L_0}\right)^2 \frac{1}{\rho} \left[\frac{\partial \mathbf{J}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{J}\right]$$

Generalized Ohm's Law

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- Resistivity
- Hall effect
- Electron pressure
- Electron inertia

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J is the electric current given by Ampère's law

$$\mathbf{J} = \nabla \times \mathbf{B}$$

X.Qian, J.Bablás, A. Bhattacharjee, H.Yang (2009)

General Induction Equation

General Induction Equation

- Faraday's law

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

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- Ampère's law
- $\overleftrightarrow{p}$ isotropic.

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Are combined to obtain

$$\frac{\partial}{\partial t}\left[\mathbf{B} + \left(\frac{\delta_e}{L_0}\right)^2 \frac{1}{\rho}\nabla \times (\nabla \times \mathbf{B})\right] = \nabla \times (\mathbf{u} \times \mathbf{B}) - \eta \nabla \times (\nabla \times \mathbf{B})$$

$$- \left(\frac{\delta_e}{L_0}\right)^2 \frac{1}{\rho}\nabla \times ((\mathbf{u} \cdot \nabla)(\nabla \times \mathbf{B})) - \frac{\delta_i}{L_0}\frac{1}{\rho}\nabla \times ((\nabla \times \mathbf{B}) \times \mathbf{B})$$

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This equation preserve the divergence of the magnetic field

$$\frac{d}{dt}(\nabla \cdot \mathbf{B}) = 0$$

Symmetrized Equation

Using the identity

$$\nabla \times (\mathbf{u} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{u} - \mathbf{B}(\nabla \cdot \mathbf{u}) + \mathbf{u}(\nabla \cdot \mathbf{B}) - (\mathbf{u} \cdot \nabla)\mathbf{B}$$

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Since the magnetic field is solenoidal $\nabla \cdot \mathbf{B} = 0$ we subtract $\mathbf{u}(\nabla \cdot \mathbf{B})$ to the right side of the equation

$$\frac{\partial}{\partial t}\left[\mathbf{B} + \left(\frac{\delta_e}{L_0}\right)^2 \nabla \times (\nabla \times \mathbf{B})\right] =$$
$$(\mathbf{B} \cdot \nabla)\mathbf{u} - \mathbf{B}(\nabla \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\mathbf{B} - \eta \nabla \times (\nabla \times \mathbf{B})$$
$$- \left(\frac{\delta_e}{L_0}\right)^2 \frac{1}{\rho} \nabla \times ((\mathbf{u} \cdot \nabla)(\nabla \times \mathbf{B})) - \frac{\delta_i}{L_0}\frac{1}{\rho} \nabla \times ((\nabla \times \mathbf{B}) \times \mathbf{B}) \quad (1)$$

Boundary Condition

$\Omega \subset \mathbb{R}^3$ is a smooth domain
$\partial\Omega_{in} = \{x \in \partial\Omega | \mathbf{n} \cdot \mathbf{u} < 0\}$ is the inflow boundary.

- Natural BC

$$\mathbf{B} \times \mathbf{n} = 0 \quad \text{on } \partial\Omega \backslash \partial\Omega_{in} \tag{2}$$

- Inflow BC

$$\begin{aligned} \mathbf{B} &= 0 \quad \text{on } \partial\Omega_{in} \\ \mathbf{J} &= 0 \quad \text{on } \partial\Omega_{in} \end{aligned} \tag{3}$$

Estimates

Theorem

For $\mathbf{u} \in C^2(\Omega)$ *and* $\mathbf{B}$ *solution of (**??**) satisfying (**??**) and (**??**), then this estimate holds*

$$\frac{d}{dt}\left(\|\mathbf{B}\|^2_{L^2(\Omega)} + \left(\frac{\delta_e}{L_0}\right)^2 \frac{1}{\rho}\|\nabla \times \mathbf{B}\|^2_{L^2(\Omega)}\right) \leq C_1\left(\|\mathbf{B}\|^2_{L^2(\Omega)} + \left(\frac{\delta_e}{L_0}\right)^2 \frac{1}{\rho}\|\nabla \times \mathbf{B}\|^2_{L^2(\Omega)}\right) \tag{4}$$

with C_1 *a constant that depend on* $\mathbf{u}$ *and its derivative only.*

Additional Boundary condition

$$\nabla \cdot \mathbf{B} = 0 \quad \text{on } \partial\Omega_{in} \tag{5}$$

Theorem

For $\mathbf{u} \in C^2(\Omega)$ *and* $\mathbf{B}$ *solution of (**??**) satisfying (**??**),then this estimate holds*

$$\frac{d}{dt}\|\nabla \cdot \mathbf{B}\|_{L^2(\Omega)} \leq C_2 \|\nabla \cdot \mathbf{B}\|_{L^2(\Omega)} \tag{6}$$

with C_2 *a constant that depend on* $\mathbf{u}$ *and its derivative only.*

Remark

If the solution satisfy all the three boundary conditions, then $\mathbf{B} \in H^1(\Omega)$

Numerical Formulation

Semi-discrete formulation

$$\frac{d}{dt}\left(\hat{\mathbf{B}}(t) + \mathbf{D} \times \mathbf{D} \times \hat{\mathbf{B}}(t)\right) = R(\hat{\mathbf{B}}(t), \hat{\mathbf{u}})$$

where $\hat{\mathbf{B}}$ and $\hat{\mathbf{u}}$ are grid functions.

- Discretize the space with $\mathbf{D}\times$.
  Resulting system of ODE satisfies similar estimates as the one we have for the continuous case.

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- Discretize the space with $\mathbf{D}\times$.
  Resulting system of ODE satisfies similar estimates as the one we have for the continuous case.
- Integrate on time with Runge Kutta Method
  We have to compute a matrix inversion!

Discrete Operator

- One dimensional D_x with summation by parts

$$D_x = P_x^{-1} Q \quad Q^\top + Q = \mathrm{diag}(-1, 0, \cdots, 0, 1)$$

P_x diagonal positive definite matrix.
$\Rightarrow (D_x \hat{v}, \hat{w})_{P_x} = -(\hat{v}, D_x \hat{w})_{P_x} + v_N w_N - v_0 w_0$.

- Multidimensional operators

$$\mathfrak{d}_x = D_x \otimes I_y \otimes I_z$$

Summation by Parts

Writing the differential operator as

$$\mathbf{D} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$$

Lemma

$$(\hat{v}, \mathbf{D}\hat{w})_{\mathbf{P}} = -(\mathbf{D}\hat{v}, \hat{w})_{\mathbf{P}} + B.T$$
$$(\hat{\mathbf{v}}, \mathbf{D} \times \hat{\mathbf{w}})_{\mathbf{P}} = (\mathbf{D} \times \hat{\mathbf{v}}, \hat{\mathbf{w}})_{\mathbf{P}} + B.T$$
$$(\hat{\mathbf{v}}, (\hat{\mathbf{u}} \cdot \mathbf{D})\hat{\mathbf{v}}))_{\mathbf{P}} \leq C\|\hat{\mathbf{v}}\|_{\mathbf{P}} + B.T$$

Numerical Scheme

$$\frac{\partial}{\partial t}\left[\hat{\mathbf{B}} + \left(\frac{\delta_e}{L_0}\right)^2 \mathbf{D}\times(\mathbf{D}\times\hat{\mathbf{B}})\right] = \mathcal{ADV}(\hat{\mathbf{u}},\hat{\mathbf{B}}) - \eta\mathbf{D}\times(\mathbf{D}\times\hat{\mathbf{B}})$$
$$- \left(\frac{\delta_e}{L_0}\right)^2 \frac{1}{\rho}\mathbf{D}\times((\hat{\mathbf{v}}\cdot\mathbf{D})\hat{\mathbf{B}}) - \frac{\delta_i}{L_0}\frac{1}{\rho}\mathbf{D}\times\left((\mathbf{D}\times\hat{\mathbf{B}})\times\hat{\mathbf{B}}\right) \qquad (7)$$

$\hat{\mathbf{B}}$ decays to zero st infinity $\Rightarrow$ neglect boundary terms. Assuming that $\mathcal{ADV}$ satisfy

Lemma

$$(\hat{\mathbf{B}}, \mathcal{ADV}(\hat{\mathbf{u}},\hat{\mathbf{B}}))_{\mathbf{P}} \leq C\|\hat{\mathbf{B}}\|^2_{\mathbf{P}} \qquad (8)$$

Theorem

*Solution of (**??**) with condition (**??**), satisfy*

$$\frac{d}{dt}\left(\|\hat{\mathbf{B}}\|_{\mathbf{P}}^2 + \left(\frac{\delta_e}{L_0}\right)^2 \frac{1}{\rho}\|\mathbf{D}\times\hat{\mathbf{B}}\|_{\mathbf{P}}^2\right)$$
$$\leq C_1\left(\|\hat{\mathbf{B}}\|_{\mathbf{P}}^2 + \left(\frac{\delta_e}{L_0}\right)^2 \frac{1}{\rho}\|\mathbf{D}\times\hat{\mathbf{B}}\|_{\mathbf{P}}^2\right)$$

can be shown using the same technique used in the continuous case.

Approximated Chain Rule

Lemma (Mishra, Svärd)

Let $\bar{u}$ be a restriction of $u \in C^2(\Omega)$, then there is a special average operator $\bar{A}_x$, such that

$$D_x(\bar{u}w) = \bar{u}D_x(w) + \bar{u}_x\bar{A}_x(w) + \tilde{w}$$

where $\|\tilde{w}\|_{P_x} \leq C\Delta x\|w\|_{P_x}$

Average operators

$$\bar{\mathcal{A}}_x = \bar{A}_x \otimes I_y \otimes I_z$$

Symmetric Advection

$$\mathcal{ADV}(\hat{\mathbf{u}}, \hat{\mathbf{B}}) = (\bar{\mathcal{A}}(\hat{\mathbf{B}}) \cdot \mathbf{D})\hat{\mathbf{u}} - \bar{\mathcal{A}}_x(\hat{\mathbf{B}})\partial_x \hat{\mathbf{u}}^1 - \bar{\mathcal{A}}_y(\hat{\mathbf{B}})\partial_y \hat{\mathbf{u}}^2 - \bar{\mathcal{A}}_z(\hat{\mathbf{B}})\partial_z \hat{\mathbf{u}}^3 - (\hat{\mathbf{u}} \cdot \mathbf{D})\hat{\mathbf{B}} \tag{9}$$

where $\bar{\mathcal{A}}(\hat{\mathbf{B}}) = (\bar{\mathcal{A}}_x(\hat{B}^1), \bar{\mathcal{A}}_y(\hat{B}^2), \bar{\mathcal{A}}_z(\hat{B}^3))^\top$
$\Rightarrow$ (**??**) is valid

Theorem (Mishra, Svärd)

*For numerical scheme with $\mathcal{ADV}$ defined as (**??**) we have*

$$\frac{d}{dt}\|\mathbf{D} \cdot \hat{\mathbf{B}}\|_{\mathbf{P}} \leq C \left(\|\mathbf{D} \cdot \hat{\mathbf{B}}\|_{\mathbf{P}} + \|\hat{\mathbf{B}}\|_{\mathbf{P}} \right)$$

Divergence Preserving

$$\mathcal{ADV}(\hat{\mathbf{u}}, \hat{\mathbf{B}}) = \mathbf{D} \times (\hat{\mathbf{u}} \times \hat{\mathbf{B}}) \tag{10}$$

The operator $\mathfrak{d}_x$, $\mathfrak{d}_y$ and $\mathfrak{d}_z$ commute
$\Rightarrow \mathbf{D} \cdot (\mathbf{D} \times \hat{\mathbf{w}}) = 0$

Lemma

*Scheme (**??**) with (**??**) satisfy*

$$\frac{d}{dt}\mathbf{D} \cdot \hat{\mathbf{B}} = 0$$

Can we obtain (**??**) for (**??**)?

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Lemma

*Scheme (**??**) with (**??**) satisfy*

$$\frac{d}{dt}\mathbf{D} \cdot \hat{\mathbf{B}} = 0$$

Can we obtain (**??**) for (**??**)?
If $\mathbf{D} \cdot \hat{\mathbf{B}}_0 = 0$ yes, symmetrizing the problem.

Time Integration

Semi-discrete formulation

$$\frac{d}{dt}[(\mathbb{I}+\mathbb{A})\hat{\mathbf{B}}(t)] = \mathbf{R}(\hat{\mathbf{B}}(t), \hat{\mathbf{u}}(t))$$

- Large dimension

$$\dim(\hat{\mathbf{B}}) = \dim(\mathbf{R}) = 3 \times N_x \times N_y \times N_z$$

- $\mathbb{A}$ matrix: discrete curl curl operator

Time evolution has to invert this matrix, e.g. Euler method

$$\hat{\mathbf{B}}^{n+1} = \hat{\mathbf{B}}^{n} + \Delta t(\mathbb{I}+\mathbb{A})^{-1}\mathbf{R}(\hat{\mathbf{B}}(t), \hat{\mathbf{u}}(t))$$

Fast Approximated Solution: Auxilary Space

joint work with Ralf Hiptmair

Known problem for edge element formulation:

Main idea:

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$$\mathbb{P}(\ker(\mathbb{A}_{FE})) \subset \ker(\mathbb{A}_{FD})$$

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Problem:

- $\mathbb{A}_{FE}$ has a “local” kernel,
- $\mathbb{A}_{FD}$ for $\mathfrak{d}_x = D_x \otimes I_y$ has a “global” kernel!

Solution:
Averages!
For

$$\mathfrak{d}_x = D_x \otimes A_y$$

with a D a second order central difference and

$$Aw_i = \frac{w_{i+1} + 2w_i + w_{i-1}}{4}$$

in this case we have a compact kernel

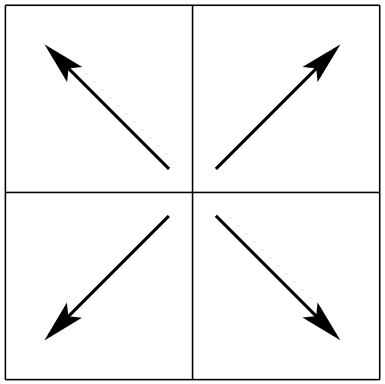

Figure: Local Star Shaped Kernel of $\mathbb{A}_{FD}$ in 2D.

Prolongation operator in 2D:

$$\mathbb{P}^\top : \text{FD space} \rightarrow \text{FE conform space}$$

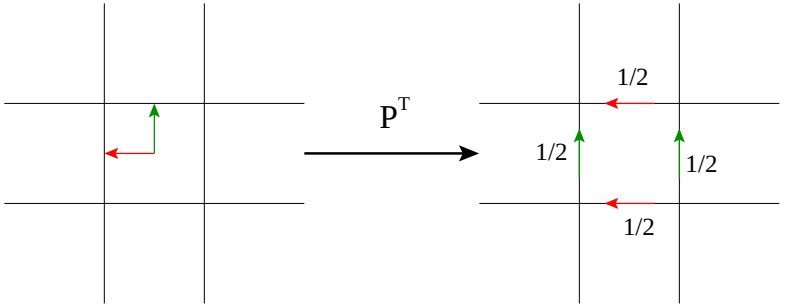

Figure: Schematic for $\mathbb{P}^\top$.

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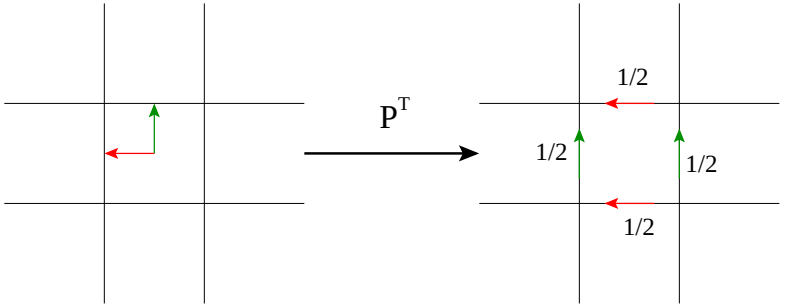


Figure: Schematic for $\mathbb{P}^\top$.

We have

$$\mathbb{P}(\ker(\mathbb{A}_{FE})) \subset \ker(\mathbb{A}_{FD})$$

New Problem: $\dim(\ker(\mathbb{A}_{FD})) \geq \dim(\ker(\mathbb{A}_{FE}))$

Additonal modes prevent fast convergence.
$\Rightarrow$ Cleaning

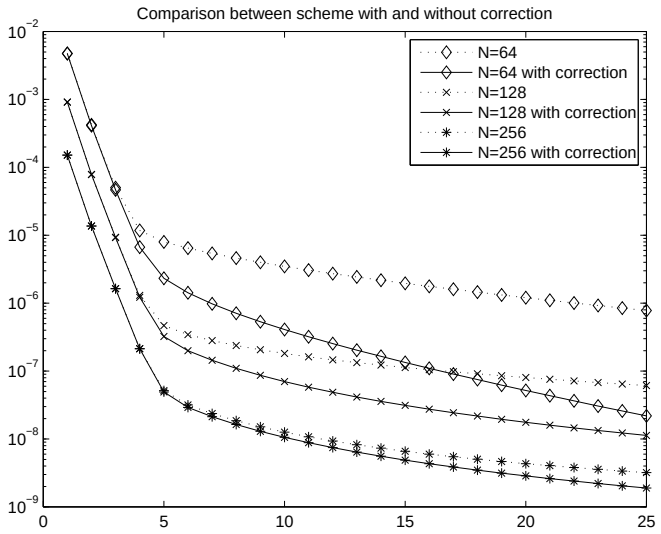


Figure: Algorithm with and without cancellation of bad modes.

Still not effective...

Still not effective...
Soultion:

- Same framework (FD): new $\hat{\mathbf{D}}$ and $\hat{\mathbf{P}}$.

OR

- New formulation DG $\Rightarrow$ more suitable space.

System of equations

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbb{U}_1 \mathbf{B} + (\mathbf{u}\nabla)\mathbf{B} = -\nabla \times \tilde{\mathbf{E}}$$

$$\mathbf{J} = \nabla \times \mathbf{B}$$

$$\tilde{\mathbf{E}} = \eta \mathbf{J} + \alpha \mathbf{J} \times \mathbf{B} + \beta\left(\frac{\partial \mathbf{J}}{\partial t} + (\mathbf{u}\nabla)\mathbf{J}\right)$$

$\mathbb{U}_1$ depends on $\frac{\partial u_i}{\partial x_j}$, $\alpha = \frac{\delta_i}{L_0 \rho}$ and $\beta = \frac{\delta_e^2}{L_0^2 \rho}$

Define

$$(v,w)_{\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h} \int_K v(x)w(x)\,dx$$

$$\langle v,w\rangle_{\text{Faces}} = \sum_{f\in\text{Faces}} \int_e v(x)w(x)\,ds$$

where $\mathcal{T}_h$ a triangulation of Ω.
Faces can be

- $\mathcal{F}_h$ set of faces in $\mathcal{T}_h$.
- $\mathcal{F}_h^{\mathcal{I}}$ set of inner faces in $\mathcal{T}_h$.
- Γ_h set of boundary faces in $\mathcal{T}_h$.
- Γ_h^+ set of outflow boundary faces in $\mathcal{T}_h$.
- Γ_h^- set of inflow faces boundary in $\mathcal{T}_h$.

To have unique valued on faces we define:

- averages $\{.\}$
- normal jumps $[\![.]\!]_N$
- tangential jumps $[\![.]\!]_T$

Find $\mathbf{B}_h, \mathbf{E}_h, \mathbf{J}_h \in \mathcal{V}_h$

$$\left(\frac{\partial \mathbf{B}_h}{\partial t} + \mathbb{U}_2 \mathbf{B}_h, \bar{\mathbf{B}}_h\right)_{\mathcal{T}_h} - \left(\mathbf{B}_h, (u\nabla)\bar{\mathbf{B}}_h\right)_{\mathcal{T}_h} + \left(\mathbf{E}_h, \nabla \times \bar{\mathbf{B}}_h\right)_{\mathcal{T}_h} +$$

$$\sum_i \langle \widehat{\mathbf{u}B^i}_h, [\![\bar{B}^i{}_h]\!]_N \rangle_{\Gamma_h^+ \cup \mathcal{F}_h^{\mathcal{I}}} - \langle \widehat{\mathbf{E}_h}, \bar{\mathbf{B}}_h \rangle_{\mathcal{F}_h} = -\langle (\mathbf{u}\cdot\mathbf{n})\mathbf{G}_1, \bar{\mathbf{B}}_h \rangle_{\Gamma_h^-}$$

$$\forall \bar{\mathbf{B}}_h \in \mathcal{V}_h$$

$$\left(\mathbf{J}_h, \bar{\mathbf{E}}_h\right)_{\mathcal{T}_h} - \left(\mathbf{B}_h, \nabla \times \bar{\mathbf{E}}_h\right)_{\mathcal{T}_h} + \langle \widehat{\mathbf{B}_h}, [\![\bar{\mathbf{E}}_h]\!]_T \rangle_{\mathcal{F}_h^{\mathcal{I}}} = 0$$

$$\forall \bar{\mathbf{E}}_h \in \mathcal{V}_h$$

$$\left(\mathbf{E}_h - (\eta - \beta(\nabla\cdot\mathbf{u}))\mathbf{J}_h - \alpha(\mathbf{J}_h \times \mathbf{B}_h) - \beta\frac{\partial \mathbf{J}_h}{\partial t}, \bar{\mathbf{J}}_h\right)_{\mathcal{T}_h} + \left(\mathbf{B}_h, (\mathbf{u}\nabla)\bar{\mathbf{J}}_h\right)_{\mathcal{T}_h}$$

$$-\beta \sum_i \langle \widehat{\mathbf{u}J^i_h}, [\![\bar{J}^i_h]\!]_N \rangle_{\Gamma_h^+ \cup \mathcal{F}_h^{\mathcal{I}}} = -\langle (\mathbf{u}\cdot\mathbf{n})\mathbf{G}_2, \bar{\mathbf{J}}_h \rangle_{\Gamma_h^-}$$

$$\forall \bar{\mathbf{J}}_h \in \mathcal{V}_h$$

DG Fluxes

- Upwind

$$\widehat{\mathbf{u}B_h^i} = \{\mathbf{u}B_h^i\} + \mathfrak{c}[\![B_h^i]\!]_N$$
$$\widehat{\mathbf{u}J_h^i} = \{\mathbf{u}J_h^i\} + \mathfrak{c}[\![J_h^i]\!]_N$$

with $\mathfrak{c} = |\mathbf{nu}|/2$.

- LDG

$$\widehat{\mathbf{B}_h} = \{\mathbf{B}_h\} + \mathfrak{b}[\![\mathbf{B}_h]\!]_T$$
$$\widehat{\mathbf{E}_h} = \begin{cases} \{\mathbf{E}_h\} - \mathfrak{b}[\![\mathbf{E}_h]\!]_T + \mathfrak{a}[\![\mathbf{B}_h]\!]_T & \text{internal edges} \\ \{\mathbf{E}_h\} + \mathfrak{a}[\![\mathbf{B}_h]\!]_T & \text{boundary edges} \end{cases}$$

With this Fluxes $\Rightarrow$ Energy estimate.

Knowing $\mathbf{B}_h^n, \mathbf{E}_h^n, \mathbf{J}_h^n$ find $\mathbf{B}_h^{n+1}, \mathbf{E}_h^{n+1}, \mathbf{J}_h^{n+1} \in \mathcal{V}_h$

$$\frac{1}{\Delta t}\left(\mathbf{B}_h^{n+1} - \mathbf{B}_h^n, \bar{\mathbf{B}}_h\right)_{\mathcal{T}_h} + \left(\mathbb{U}_2\mathbf{B}_h^n, \bar{\mathbf{B}}_h\right)_{\mathcal{T}_h} - \left(\mathbf{B}_h^n, (u\nabla)\bar{\mathbf{B}}_h\right)_{\mathcal{T}_h} +$$

$$\left(\mathbf{E}_h^{n+1}, \nabla \times \bar{\mathbf{B}}_h\right)_{\mathcal{T}_h} + \sum_i \langle \widehat{\mathbf{u}(B^i)^n}_h, [\![\bar{B}^i{}_h]\!]_N \rangle_{\Gamma_h^+ \cup \mathcal{F}_h^{\mathcal{I}}} - \langle \widehat{\mathbf{E}_h^{n+1}}, \bar{\mathbf{B}}_h \rangle_{\mathcal{F}_h}$$

$$= -\langle (\mathbf{u}\cdot\mathbf{n})\mathbf{G}_1(t^n), \bar{\mathbf{B}}_h \rangle_{\Gamma_h^-} \qquad \forall\, \bar{\mathbf{E}}_h \in \mathcal{V}_h$$

$$\left(\mathbf{J}_h^n, \bar{\mathbf{E}}_h\right)_{\mathcal{T}_h} - \left(\mathbf{B}_h^n, \nabla \times \bar{\mathbf{E}}_h\right)_{\mathcal{T}_h} + \langle \widehat{\mathbf{B}_h^n}, [\![\bar{\mathbf{E}}_h^n]\!]_T \rangle_{\mathcal{F}_h^{\mathcal{I}}} = 0 \qquad \forall\, \bar{\mathbf{B}}_h \in \mathcal{V}_h$$

$$\left(\mathbf{E}^{\mathbf{n+1}}{}_h - \eta\mathbf{J}_h^{n+1} + \beta(\nabla\cdot\mathbf{u}))\mathbf{J}_h^n - \alpha(\mathbf{J}_h^{n+1} \times \mathbf{B}_h^n), \bar{\mathbf{J}}_h\right)_{\mathcal{T}_h}$$

$$-\frac{\beta}{\Delta t}\left(\mathbf{J}_h^{n+1} - \mathbf{J}_h^n, \bar{\mathbf{J}}_h\right)_{\mathcal{T}_h} + \left(\mathbf{B}_h^n, (\mathbf{u}\nabla)\bar{\mathbf{J}}_h\right)_{\mathcal{T}_h} - \beta\sum_i \langle \widehat{\mathbf{u}(J^i)_h^n}, [\![\bar{J}_h^i]\!]_N \rangle_{\Gamma_h^+ \cup \mathcal{F}_h^{\mathcal{I}}}$$

$$= -\langle (\mathbf{u}\cdot\mathbf{n})\mathbf{G}_2(t^n), \bar{\mathbf{J}}_h \rangle_{\Gamma_h^-} \qquad \forall\, \bar{\mathbf{J}}_h \in \mathcal{V}_h$$

Matrix formulation

$$\underbrace{\left(\gamma_1^n \mathbb{M} + \gamma_2^n \mathbb{A}_{DG}\right)}_{\mathbb{C}_{DG}^n} \hat{\mathbf{B}}^{n+1} = \mathbf{R}^n$$

Algorithm

$\mathbf{B}^{n+1} \leftarrow \mathbf{B}^n$
for $i \leq N_{\text{iter}}$ **do**
  $\mathbf{r} \leftarrow \mathbf{R}^n - \mathbb{C}_{DG}^n \mathbf{B}^{n+1}$
  $\mathbf{c} \leftarrow \mathbb{L}_{DG}^n \mathbf{c} = \mathbf{r}$
  $\mathbf{B}^{n+1} \leftarrow \mathbf{B}^{n+1} + \mathbf{c}$
  $\mathbf{r} \leftarrow \mathbf{R}^n - \mathbb{C}_{DG}^n \mathbf{B}^{n+1}$
  $\rho \leftarrow \mathbb{P}^\top \mathbf{r}$
  $\gamma \leftarrow \mathbb{C}_{FE}^n \gamma = \rho$
  $\mathbf{c} \leftarrow \mathbb{P}\gamma$
  $\mathbf{B}^{n+1} \leftarrow \mathbf{B}^{n+1} + \hat{\mathbf{c}}$
  $i \leftarrow i + 1$
end for

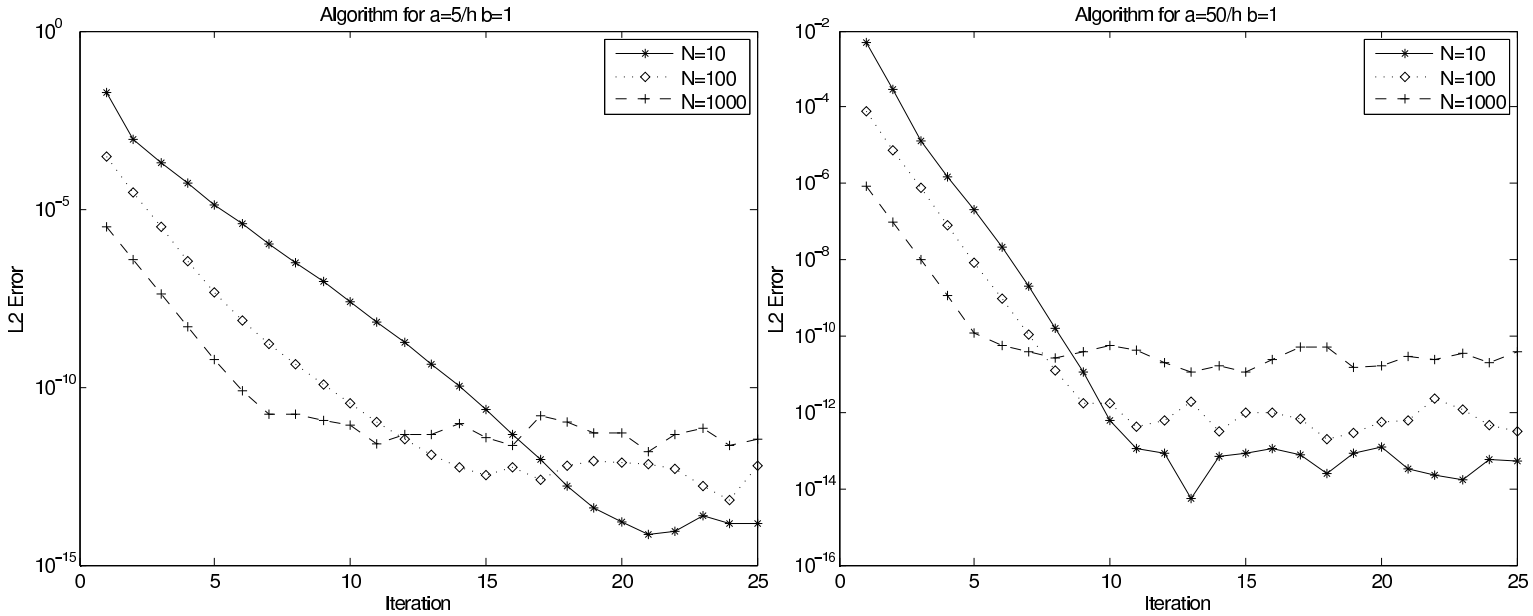
1D Model

$$\partial_t b + v\partial_x b - \eta\partial_{xx} b - \beta(\partial_{xxt} b + \partial_x(u\partial_{xx} b)) = 0$$

Auxiliary variables

$$\begin{aligned} \partial_t b + u\partial_x b - \partial_x e &= 0 \\ j &= \partial_x b \\ e &= \eta j + \beta(\partial_t j + u\partial_x j) \end{aligned}$$

Preconditioner



Conclusion

- The solution of the induction equation with Hall term is in $H^1(\Omega)$.
- We can build symmetric and divergence preserving methods.
- The spatially discretized systems possess similar estimates as continuous one.
- $\Rightarrow$ stability granted for exact-time evolution of the discrete system.
- Preconditioner for time evolution.
- DG Formulation.

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Thank You!