PRO*DOC WORKSHOP

Approximation by waves

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Model problem

Homogeneous Helmholtz equation with impedance boundary conditions:

$$\begin{cases} -\Delta u - \omega^2 u = 0 & \text{in } \Omega, \\ \nabla u \cdot \mathbf{n} + i\omega u = g & \text{on } \partial\Omega, \end{cases}$$

$\Omega \subset \mathbb{R}^N$, $N = 2, 3$, bounded polyhedral domain, $\quad g \in L^2(\partial\Omega)$, wavenumber $\omega > 0$.

With large ω, standard finite elements are affected by the numerical dispersion and the pollution effect:

special FEM are required.

Trefftz methods for Helmholtz

The Trefftz methods use basis functions that are solutions of the PDE under examination in each element.

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Homogeneous Helmholtz equation: $-\Delta u - \omega^2 u = 0$

- plane wave basis functions (UWVF, PWDG, DEM, VTCR,...)

$$x \mapsto e^{i\omega d\cdot x}, \qquad |d| = 1;$$

- circular / spherical / corner wave basis functions (least-squares Trefftz,...);
- singular / fundamental basis functions (MFS,...);
- ...

The best approximation estimate

The analysis of plane waves Trefftz methods requires best approximation estimates:

$$-\Delta u - \omega^2 u = 0 \qquad \text{in } D, \qquad \operatorname{diam} D = h, \qquad p \in \mathbb{N},$$

$$\inf_{\alpha \in \mathbb{C}^p} \left\| u - \sum_{k=1}^{p} \alpha_k e^{i\omega d_k \cdot x} \right\|_{H^j(D)} \leq C\, \epsilon(h,p)\ \|u\|_{H^{K+1}(D)},$$

with $\epsilon(h,p) \xrightarrow[p\to\infty]{h\to 0} 0$.

The plan

Only few results are available (Cessenat and Després 1998, Melenk 1995), improvement and generalization are needed.

Our goals:

- estimates for plane and circular/spherical waves;
- estimates both in h and p;
- estimates in 2 and 3 dimensions;
- explicit dependence of the bounds on ω.

Outline

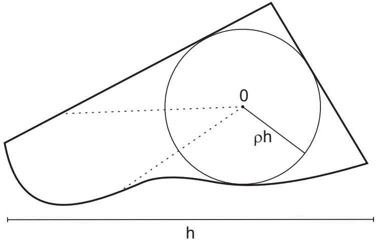
- Vekua theory;
- approximation by generalized harmonic polynomials;
- approximation by plane waves;
- approximation for Maxwell equations (only some ideas!).

Part I

Vekua theory

The Vekua theory in N dimensions

$D \subset \mathbb{R}^N, \ N \geq 2$,
star-shaped.



Given $\omega > 0$, define two continuous functions:

$$M_1, M_2 : D \times [0,1] \to \mathbb{R}$$

$$M_1(x,t) = -\frac{\omega|x|}{2} \frac{\sqrt{t}^{N-2}}{\sqrt{1-t}} J_1(\omega|x|\sqrt{1-t}),$$

$$M_2(x,t) = -\frac{i\omega|x|}{2} \frac{\sqrt{t}^{N-3}}{\sqrt{1-t}} J_1(i\omega|x|\sqrt{t(1-t)}).$$

J_1 is the ordinary Bessel function of the first kind and order 1.

The Vekua operators

$$V_1, V_2 : C(D) \to C(D),$$

$$V_j[\phi](x) := \phi(x) + \int_0^1 M_j(x,t)\phi(tx)\,\mathrm{d}t, \qquad a.e.\ x \in D,\ j = 1, 2.$$

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Short Vekua's story:

- Vekua 1942, Helmholtz equation in N dimensions, no proofs;
- Vekua 1948 (translated in 1967), elliptic equations in 2 dimensions, one page for N-dimensional Helmholtz;
- Henrici 1957, elliptic equations only in 2 dimensions.

$\to$ N-dimensional version almost forgotten.

The properties of Vekua operators

1. $$V_2 = (V_1)^{-1}$$

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2. $$(-\Delta - \omega^2)\, u = 0 \qquad \Longleftrightarrow \qquad \Delta\, V_2[u] = 0$$

Main idea of Vekua theory:

$$\text{Helmholtz solutions} \quad \underset{V_2}{\overset{V_1}{\rightleftarrows}} \quad \text{Harmonic functions}$$

The continuity of Vekua operators

Weighted Sobolev norms:

$$\|u\|_{j,\omega,D} := \left(\sum_{k=0}^{j} \omega^{2(j-k)} \, |u|_{k,D}^2 \right)^{\frac{1}{2}}, \qquad \omega > 0, \ j \in \mathbb{N}.$$

$\Delta\phi = 0$, $(-\Delta - \omega^2)u = 0$:

$$\|V_1[\phi]\|_{j,\omega,D} \leq C_1(N,j,\rho) \left(1 + (\omega h)^2\right) \|\phi\|_{j,\omega,D} \qquad j \geq 0 \,,$$

$$\|V_2[u]\|_{j,\omega,D} \leq C_2(N,j,\rho) \left(1 + (\omega h)^4\right) e^{\frac{3}{4}\omega h} \|u\|_{j,\omega,D} \qquad j \geq 1 \,,$$

$$N = 2, 3, \quad u, \ \phi \in H^j(D).$$

The interior estimates

Key ingredients for the proof of the continuity of V_1 and V_2 are the interior estimates.

- For harmonic functions, these are well-known:

$$\Delta\phi = 0 \quad \Rightarrow \quad |\phi(x)|^2 \leq \frac{1}{R^N\,|B_1|}\,\|\phi\|^2_{0,B_{(x,R)}}\,.$$

- For Helmholtz solutions, $N = 2,\,3$, we can prove estimates that are explicit in ω:

$$-\Delta u - \omega^2 u = 0$$
$$\Downarrow$$
$$|u(x)| \leq C\,R^{-\frac{N}{2}}\,(1+\omega^2R^2)\,\left(\|u\|_{0,B_{(x,R)}} + R\,\|\nabla u\|_{0,B_{(x,R)}}\right),$$
$$|\nabla u(x)| \leq C\,R^{-\frac{N}{2}}\,\left(\omega^2 R\,\|u\|_{0,B_{(x,R)}} + (1+\omega^2R^2)\,\|\nabla u\|_{0,B_{(x,R)}}\right).$$

Part II

Approximation by GHPs

The approximation by GHPs

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$$\downarrow V_2$$

$V_2[u]$ is harmonic $\quad\Longrightarrow\quad$ can be approximated
by harmonic polynomials
(Bramble-Hilbert, complex analysis techniques, . . .),

The approximation by GHPs

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(Bramble-Hilbert, complex analysis techniques, . . .),

$$\downarrow V_1$$

u can be approximated by

$$V_1\left[\begin{array}{c}\text{harmonic}\\ \text{polynomials}\end{array}\right] =: \begin{array}{c}\text{generalized}\\ \text{harmonic}\\ \text{polynomials}\end{array} \qquad \text{(GHPs).}$$

Generalized harmonic polynomials

In 2D, the GHPs are circular waves:

$$Q(x) = e^{il\psi} J_l(\omega r),$$

in polar coordinates $x = re^{i\psi}$, $l \in \mathbb{Z}$,

J_l = Bessel functions.

In 3D, the GHPs are spherical waves:

$$Q(x) = Y_{l,m}\left(\frac{x}{|x|}\right) j_l(\omega|x|),$$

$0 \le |m| \le l \in \mathbb{N}$,

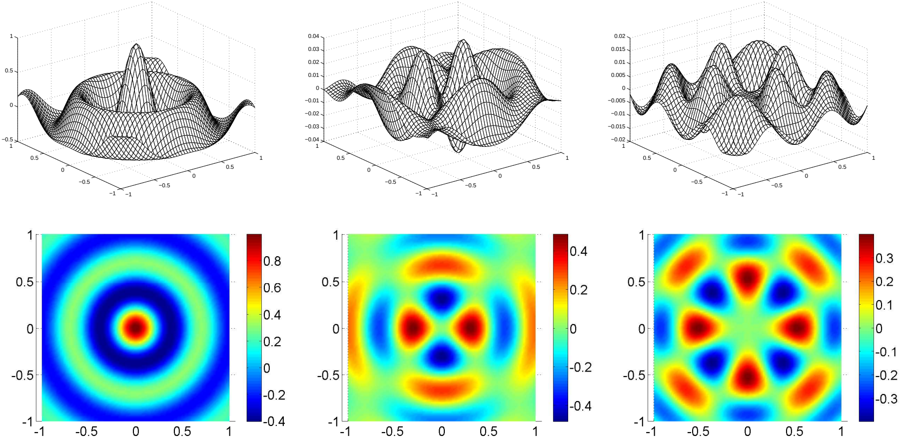
$Y_{l,m}$ = spherical harmonics,
j_l = spherical Bessel functions.

Both belong to the family of the Herglotz functions.

The GHPs are also called Fourier-Bessel functions.

Generalized harmonic polynomials

Real part of GHPs in $[-1,1]^2$: $V_1[z^l]$, $l = 0, 2, 4$, $\omega = 10$



The approximation by GHPs: h-convergence

$$\inf_{P \in \left\{\substack{\text{harmonic} \\ \text{polynomials} \\ \text{of degree } \leq L}\right\}} \|u - V_1[P]\|_{j,\omega,D} \leq C \inf_P \|V_2[u] - P\|_{j,\omega,D} \qquad \text{contin. of } V_1,$$

$$\leq C\, h^{k+1-j}\, \epsilon(L)\, \|V_2[u]\|_{k+1,\omega,D} \qquad \text{harmonic approx. results,}$$

$$\leq C\, h^{k+1-j}\, \epsilon(L)\, \|u\|_{k+1,\omega,D} \qquad \text{contin. of } V_2.$$

For the h-convergence it is enough to use Bramble-Hilbert theorem: it provides a harmonic polynomial!

The constant C depends on ωh, not on ω alone:

$$C = C \cdot (1 + \omega h)^{j+6} e^{\frac{3}{4}\omega h}.$$

Harmonic approximation: p-convergence, 2D

In 2 dimensions, if D satisfies the 'exterior cone condition', then:

$$\epsilon(L) = \left(\frac{\log(L+2)}{L+2}\right)^{\lambda(k+1-j)}.$$

If D is convex, $\lambda = 1$. Otherwise $\lambda = \min \frac{\text{re-entrant corner of } D}{\pi}$.

Sharp p-estimate! (MELENK)

In 2D we can use complex analysis:

- conformal mappings $B_1 \leftrightarrow D$;
- ϕ harmonic and real $\rightarrow$ $\phi =$ Re(Φ holomorphic);
- complex interpolation on B_1;
- every (complex) polynomial is harmonic;
- ...

Harmonic approximation: p-convergence, N-D

An analogous result holds in N dimensions:

$$\epsilon(L) = L^{-\lambda(k+1-j)},$$

where $\lambda > 0$ is a geometric unknown parameter.

Follows from:

- an exponential approximation result for compact subsets by (BAGBY, BOS AND LEVENBERG 1996) in L^∞-norm;
- harmonic dilation and deformation technique.

If u is the restriction of a solution in a larger domain (2 or 3D), the convergence in L is exponential.

The approximation by GHPs: $h\&p$-convergence

Harmonic polynomial approximation:

2-3D	h-conv.	harmonic Bramble-Hilbert: easy
2D	p-conv.	sharp estimate by Melenk: complex analysis
3D	p-conv.	new estimates, order of convergence not explicit

$\downarrow$ Vekua continuity

If $-\Delta u - \omega^2 u = 0, \quad N = 2, 3, \quad 0 \le j \le k \le L$

$$\inf_{Q \in \left\{ \substack{\text{GHPs} \\ \text{of degree } \le L} \right\}} \|u - Q\|_{j,\omega,D} \le C(\omega h)\, h^{k+1-j}\, L^{-\lambda(k+1-j)}\, \|u\|_{k+1,\omega,D},$$

The rate $\lambda > 0$ depends only on the shape of D.
If it is convex and 2D, then $\lambda = 1 - \varepsilon$.

Best approx. estimate for spherical waves Trefftz methods!

Part III

Approximation by plane waves

The approximation of GHPs by plane waves

Link between plane waves and circular/spherical waves:
Jacobi-Anger expansion

2D $$e^{iz\cos\theta} = \sum_{l\in\mathbb{Z}} i^l J_l(z)\, e^{il\theta}, \qquad z\in\mathbb{C},\ \theta\in\mathbb{R},$$

3D $$\underbrace{e^{ir\xi\cdot\eta}}_{\text{plane wave}} = 4\pi \sum_{l\geq 0}\sum_{m=-l}^{l} i^l\, \underbrace{j_l(r)\, Y_{l,m}(\xi)}_{\text{GHP}}\, \overline{Y_{l,m}(\eta)}, \qquad \xi,\,\eta\in S^2,\ r\geq 0.$$

We need the other way round:

GHP = linear combination of plane waves

$\rightarrow$ truncation of J-A expansion,
$\rightarrow$ solution of a linear system,
$\rightarrow$ residual estimates.

The approximation of GHPs by plane waves: 2D

$$\text{GHP} - \sum_{k=-q}^{q} \alpha_k \, e^{i\omega x \cdot d_k} \qquad (\, x = |x| e^{i\psi}, \; d_k = (\cos\theta_k, \sin\theta_k) \,)$$

$$= \underbrace{\sum_{l=-L}^{L} a_l \, J_l(\omega|x|) \, e^{il\psi}}_{V[P],\ \text{degree } L} - \underbrace{\sum_{l\in\mathbb{Z}} J_l(\omega|x|) \, e^{il\psi} \; i^l \sum_{k=-q}^{q} \alpha_k \, e^{-il\theta_k}}_{\text{Jacobi-Anger}}$$

$$= - \sum_{|l|>q} i^l \, J_l(\omega|x|) \, e^{il\psi} \sum_{k=-q}^{q} \alpha_k \, e^{-il\theta_k},$$

where the vector α_k is solution of a Vandermonde linear system:

$$\{e^{-i\,l\,\theta_k}\}_{l,k} \cdot \alpha_k = i^{-l} a_l$$

- bound on the inverse matrix;
- control on the minimal angular distance between plane waves directions (non equispaced case).

The approximation of GHPs by plane waves: 2D

Given a harmonic polynomial P of degree at most $L(\geq K)$, there exists $\alpha \in \mathbb{C}^{2q+1}$ such that

$$\left\| V_1[P] - \sum_{k=1}^{2q+1} \alpha_k \, e^{i\omega x \cdot d_k} \right\|_{L^\infty(B_{2h})} \leq C_{(\rho,L,\omega h)} \, h^{K-1} \, \left(\frac{c_0 \, (\omega h)^2}{q+1} \right)^{\frac{q+1}{2}} \|P\|_{K,\omega,D} \, .$$

The convergence is faster than exponential in q.

The constants C and c_0 can be made completely explicit.

$\left.\begin{array}{l}\text{Cauchy estimates}\\ \text{Vekua continuity}\end{array}\right\} \to$ bound in Sobolev norms.

The choice of the directions in 3D

3D Jacobi-Anger gives the matrix $\{M\}_{l,m;k} = Y_{l,m}(d_k)$
that depends on the choice of the directions.

Problem: an upper bound on $\|M^{-1}\|$ is needed but M is not even always invertible!

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Solution:

- there exists an optimal choice s.t. $\|M^{-1}\|_1 \leq 2\sqrt{\pi}\, p$;
- it corresponds to the extremal systems of Sloan and Womersley for quadrature on S^2;
- some simple choices of points give good result, heuristic: d_k have to be as "equispaced" as possible.

With this choice $\rightarrow$ analogous results as in 2D.

The final approximation by plane waves

$-\Delta u - \omega^2 u = 0$

$\downarrow$

GHPs

$\downarrow$

Plane waves

Vekua theory: algebraic in h and p,

Jacobi-Anger: algebraic in h,
$>$ exponential in p.

Final estimate

$$\inf_{\alpha \in \mathbb{C}^p} \left\| u - \sum_{k=1}^{p} \alpha_k e^{i\omega x \cdot d_k} \right\|_{j,\omega,D} \leq C\, h^{K+1-j} q^{-\lambda(K+1-j)} \|u\|_{K+1,\omega,D}$$

In 2D:	$p = 2q + 1$,	$\lambda(D)$	explicit,	$\forall\, d_k$.
In 3D:	$p = (q+1)^2$,	$\lambda(D)$	unknown,	special d_k.

Approximation for Helmholtz eq.: conclusions

What we have proved:

- hp-estimates for circular/spherical and plane waves in 2D and 3D,
- all the constants are explicit in ωh,
- all the orders in h are sharp.

Open problems / work in progress:

- explicit order of convergence in p in 3D convex domains,
- Vekua theory and approximation for Maxwell equation.

Part IV

Approximation for Maxwell equations

Maxwell equation

The vector field $\mathbf{u}$ is solution of Maxwell equations

$$\operatorname{curl}\operatorname{curl}\mathbf{u} - \omega^2\mathbf{u} = 0$$

if and only if

$$\begin{cases} -\Delta\mathbf{u}_j - \omega^2\mathbf{u}_j = 0, \qquad j = 1, 2, 3, \\ \operatorname{div}\mathbf{u} = 0. \end{cases}$$

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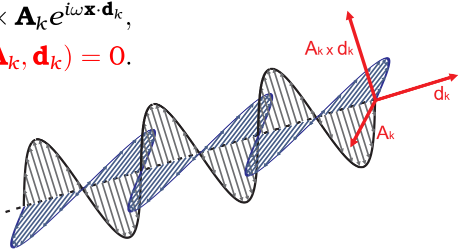
$\begin{cases} p \text{ directions} \\ 3p \text{ plane waves} \end{cases} \longrightarrow$ same approximation as for Helmholtz, non-Trefftz functions!

Maxwell plane waves

Basis of Maxwell plane wave functions:

$$\mathbf{A}_k e^{i\omega \mathbf{x}\cdot\mathbf{d}_k}, \qquad \mathbf{d}_k \times \mathbf{A}_k e^{i\omega \mathbf{x}\cdot\mathbf{d}_k},$$

$$|\mathbf{A}_k| = |\mathbf{d}_k| = 1,\ (\mathbf{A}_k, \mathbf{d}_k) = 0.$$



Goal: prove convergence using $2p$ plane waves and preserving the Trefftz property.

P.W. approximation: lazy approach

1 $\mathbf{u}$ Maxwell $\Rightarrow$ $\operatorname{curl}\mathbf{u}$ Maxwell $\Rightarrow$ $\operatorname{curl}\mathbf{u}$ Helmoltz

$$\left\| \operatorname{curl}\mathbf{u} - \begin{matrix}\text{Helmholtz}\\ \text{vector p.w.}\end{matrix} \right\|_{j,\omega,D} \le C(hq^{-\lambda})^{K+1-j} \left\|\operatorname{curl}\mathbf{u}\right\|_{K+1,\omega,D}$$

2 With $j \geq 1$, apply curl and reduce j (bad!):

$$\left\| \operatorname{curl}\operatorname{curl}\mathbf{u} - \operatorname{curl}\begin{bmatrix}\text{Helmholtz}\\ \text{vector p.w.}\end{bmatrix} \right\|_{j-1,\omega,D} \le C(hq^{-\lambda})^{K+1-j} \left\|\operatorname{curl}\mathbf{u}\right\|_{K+1,\omega,D}$$

$$\Downarrow$$

3 $$\left\| \omega^2\mathbf{u} - \text{Maxwell p.w.} \right\|_{j-1,\omega,D} \le C(hq^{-\lambda})^{K+1-j} \left\|\operatorname{curl}\mathbf{u}\right\|_{K+1,\omega,D}$$

Mismatch between Sobolev indices and convergence order:

not sharp!

Vector S.H.: welcome to the jungle!

Two different orthogonal basis for $\left(L^2(S^2)\right)^3$:

$$\begin{cases} \mathbf{Y}_{l,m}(\mathbf{x}) = Y_{l,m}(\mathbf{x})\ \dfrac{\mathbf{x}}{|\mathbf{x}|} \\ \mathbf{\Psi}_{l,m}(\mathbf{x}) = |\mathbf{x}|\ \nabla Y_{l,m}(\mathbf{x}) \\ \mathbf{\Phi}_{l,m}(\mathbf{x}) = \mathbf{x} \times \nabla Y_{l,m}(\mathbf{x}) \end{cases}$$

$$\begin{cases} \mathbf{I}_{l,m}(\mathbf{x}) = (l+1)\ \mathbf{Y}_{l+1,m}(\mathbf{x}) + \mathbf{\Psi}_{l+1,m}(\mathbf{x}) \\ \mathbf{T}_{l,m}(\mathbf{x}) = -\mathbf{\Phi}_{l,m}(\mathbf{x}) \\ \mathbf{N}_{l,m}(\mathbf{x}) = l\ \mathbf{Y}_{l-1,m}(\mathbf{x}) - \mathbf{\Psi}_{l-1,m}(\mathbf{x}) \end{cases}$$

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$\boldsymbol{\Phi}_{l,m}, \boldsymbol{\Psi}_{l,m}$ tangential to S^2, $\mathbf{Y}_{l,m}$ orthogonal,

$\mathbf{I}_{l,m}, \mathbf{T}_{l,m}, \mathbf{N}_{l,m}$ are traces of l-homogeneous harmonic vector polynomials.

Maxwell-GHPs

Herglotz functions theory $\rightarrow$ basis of Maxwell-GHPs:

$$\begin{cases} \mathbf{b}^1_{l,m}(\mathbf{x}) = j_l(\omega r)\, \mathbf{\Phi}_{l,m} \\ \mathbf{b}^2_{l,m}(\mathbf{x}) = \dfrac{l+1}{2l+1} j_{l-1}(\omega r)\, \mathbf{I}_{l-1,m} + \dfrac{l}{2l+1} j_{l+1}(\omega r)\, \mathbf{N}_{l+1,m}. \end{cases}$$

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In $\mathbf{b}^2_{l,m}$ there are two Bessel functions of different indices:

$\mathbf{V}_2[\mathbf{b}^2_{l,m}]$ is a non-homogeneous polynomial !

$\rightarrow$ approximation theory is a bit more complicated.

Approximation in h by Maxwell-GHPs

- (VERFÜRTH 1999) recursive definition of approximating polynomial $\mathbf{P}_{m,B}$ on D: orders in h;
- for harmonic vector fields $\mathbf{P}_{m,B}$ is the Taylor polynomial!
- approximate $\mathbf{P}_{m,B}$ with $\{\mathbf{V}_2[\mathbf{b}^1_{l,m}]\}_{l\leq L} \cup \{\mathbf{V}_2[\mathbf{b}^2_{l,m}]\}_{l\leq L+1}$;
- Vekua continuity $\Rightarrow$ h-estimate for Maxwell solutions:

$$\|\mathbf{u} - \mathbf{Q}_L\|_{j,\omega,D} \leq C(\omega h)\; h^{L+1-j}\;\; \|\mathbf{u}\|_{L+1,\omega,D}$$

- same order as Helmholtz, using a few more functions:

$$2(L+1)^2 - 2 + 2L + 3.$$

p-estimate impossible with this approach without any new ideas for the Helmholtz case.

Approximation in h by Maxwell plane waves

GHPs$\leftrightarrow$plane waves approximation relies on Jacobi-Anger exp.

We need a vector version of the Jacobi-Anger expansion that "preserves" Maxwell property:

$$e^{i\omega\mathbf{x}\cdot\mathbf{d}}\mathrm{Id} = 4\pi \sum_{l\geq 0}\sum_{|m|\leq l} i^l \Big(\tfrac{1}{l(l+1)}\mathbf{b}^1_{l,m}(\omega\mathbf{x})\,\overline{\mathbf{\Phi}_{l,m}(\mathbf{d})} - \tfrac{i}{l(l+1)}\,\mathbf{b}^2_{l,m}(\omega\mathbf{x})\,\overline{\mathbf{\Psi}_{l,m}(\mathbf{d})} - i\,\mathbf{b}^{\perp}_{l,m}(\omega\mathbf{x})\,\overline{\mathbf{Y}_{l,m}(\mathbf{d})} \Big)$$

($\mathbf{b}^{\perp}_{l,m}$ are the non-Maxwell vector GHPs).

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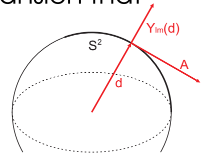

$$e^{i\omega\mathbf{x}\cdot\mathbf{d}}\quad \mathbf{A} = 4\pi\sum_{l\geq 0}\sum_{|m|\leq l} i^l\Bigg(\frac{1}{l(l+1)}\mathbf{b}^1_{l,m}(\omega\mathbf{x})\,\overline{\mathbf{\Phi}_{l,m}(\mathbf{d})}\cdot\mathbf{A}$$

$$-\frac{i}{l(l+1)}\,\mathbf{b}^2_{l,m}(\omega\mathbf{x})\,\overline{\mathbf{\Psi}_{l,m}(\mathbf{d})}\cdot\mathbf{A} - \underbrace{i\,\mathbf{b}^\perp_{l,m}(\omega\mathbf{x})\,\overline{\mathbf{Y}_{l,m}(\mathbf{d})}\cdot\mathbf{A}}_{=0 \text{ if } \mathbf{d}\cdot\mathbf{A}=0}\Bigg)$$

($\mathbf{b}^\perp_{l,m}$ are the non-Maxwell vector GHPs).

Maxwell PW = infinite linear combination of Maxwell GHPs.

Still some (technical ?) problems in the residual estimate to prove approximation in h.

Thank You!