FVM for the Two-fluid MHD Equations

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Table of contents

Plasma Flow Models

$$\frac{\partial f_\alpha}{\partial t} + \vec{v} \cdot \nabla_{\vec{x}} f_\alpha + \frac{q_\alpha}{m_\alpha}(\vec{E} + \vec{v} \times \vec{B}) \cdot \nabla_{\vec{v}} f_\alpha = C_\alpha$$

- Boltzmann equation based on statistical description.
- Describe the collective behavior of plasma species α.
- $\vec{E}$ and $\vec{B}$ are given by Maxwell's equations.
- C_α is collision operator.

Plasma Flow Models

$$\frac{\partial f_\alpha}{\partial t} + \vec{v} \cdot \nabla_{\vec{x}} f_\alpha + \frac{q_\alpha}{m_\alpha}(\vec{E} + \vec{v} \times \vec{B}) \cdot \nabla_{\vec{v}} f_\alpha = C_\alpha$$

- Boltzmann equation based on statistical description.
- Describe the collective behavior of plasma species α.
- $\vec{E}$ and $\vec{B}$ are given by Maxwell's equations.
- C_α is collision operator.
- $f_\alpha(\vec{x}, \vec{v}, t)$ depends on seven variables.
- Difficult to analyze and compute.

Moments of Boltzmann Equation

- Moments of the Boltzmann equation w.r.t velocity $\vec{v}$.
- Results in infinite chain of equations, need to truncate.
- Assumptions enforce the domain of validity.

Moments of Boltzmann Equation

- Moments of the Boltzmann equation w.r.t velocity $\vec{v}$.
- Results in infinite chain of equations, need to truncate.
- Assumptions enforce the domain of validity.
- First three moments with the assumption of local thermodynamical equilibrium, results in *Two-Fluid equations*.
- Ignoring *viscous, resistive* and *thermal diffusion* effect gives Ideal Two-Fluid equations.

Ideal Two-fluid equations:Flow equations

- Zeroth moment yields Mass conservation

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \vec{v}_\alpha) = 0,$$

with $\alpha \in \{i, e\}$.

Ideal Two-fluid equations:Flow equations

- Zeroth moment yields Mass conservation

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \vec{v}_\alpha) = 0,$$

with $\alpha \in \{i, e\}$.

- First moments, written in conservation form results in Momentum conservation

$$\frac{\partial (\rho_\alpha \vec{v}_\alpha)}{\partial t} + \nabla \cdot \left(\rho_\alpha \vec{v}_\alpha \vec{v}_\alpha^\top + p_\alpha \mathbf{I} \right) = \frac{q_\alpha}{m_\alpha} \rho_\alpha (\vec{E} + \vec{v}_\alpha \times \vec{B}).$$

Ideal Two-fluid equations:Flow equations

- Second moment give Energy conservation

$$\frac{\partial E_\alpha}{\partial t} + \nabla \cdot ((E_\alpha + p_\alpha)\vec{v}_\alpha) = r_\alpha \rho_\alpha (\vec{E} \cdot \vec{v}_\alpha).$$

- Equation of state for ideal gas,

$$E_\alpha = \frac{p_\alpha}{\gamma - 1} + \frac{1}{2}\rho_\alpha |\vec{v}_\alpha|^2$$

with $\gamma = \frac{5}{3}$ is gas constant.

Maxwell's Equation

- Closed with Perfectly Hyperbolic Maxwell's Equations (PHM).
- Allows approximate div free evolution of the magnetic field.
- Maxwell's Equation

$$\begin{aligned}
\frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} + \kappa \nabla \psi &= 0, \\
\frac{\partial \psi}{\partial t} + \kappa c^2 \nabla \cdot \vec{B} &= 0, \\
\frac{\partial \vec{E}}{\partial t} - c^2 \nabla \times \vec{B} + \xi c^2 \nabla \phi &= -\frac{1}{\varepsilon_0}\left(r_i \rho_i \vec{v}_i + r_e \rho_e \vec{v}_e\right), \\
\frac{\partial \phi}{\partial t} + \xi \nabla \cdot \vec{E} &= \frac{\xi}{\varepsilon_0}(r_i \rho_i + r_e \rho_e)
\end{aligned}$$

Application of Plasma Flows

- Geospace Environment modeling(GEM)-Magnetic Reconnection
- Electric propulsion-Hall effect thrusters.
- Control fusion-Plasma confinement.
- Circuit Breakers.
- Waste management.

Flux

- Ion-Electron flux

$$\vec{\mathbf{F}}_\alpha(\vec{\mathbf{U}}) = \begin{cases} \rho_\alpha \vec{v}_\alpha \\ \rho_\alpha \vec{v}_\alpha \vec{v}_\alpha^\top + p_\alpha \mathbf{I}, \\ (E_\alpha + p_\alpha)\vec{v}_\alpha \end{cases}$$

- Nonlinear Euler flux.
- Maxwell's Flux is Linear.

Flux

- Ion-Electron flux

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- Nonlinear Euler flux.
- Maxwell's Flux is Linear.
- Flux parts are completely split and Eqns. are coupled through source terms only.

Entropy

Lemma

Let s_i, s_e be defined as fluid entropies,

$$s_i = \log p_i - \gamma \log \rho_i \qquad s_e = \log p_e - \gamma \log \rho_e$$

Then, smooth solutions of TF Eqns. in one dimension satisfy,

$$(\rho_i s_i)_t + (\rho v_i^x s_i)_x = 0, \quad (\rho_e s_e)_t + (\rho_e v_e^x s_e)_x = 0$$

Entropy

Proof.

A simple calculation shows,

$$\rho_{i_t} + \vec{v}_i^x \rho_i + \rho_i(\vec{v}_i^x)_x = 0,$$
$$p_{i_t} + \gamma p_i(\vec{v}_i^x)x + \vec{v}_i^x p_{i_x} = 0,$$

Combining,

$$(s_i)_t + \vec{v}_i^x (s_i)_x = 0,$$

Adding mass conservation equation yields the required equality. □

Entropy

Remark

As is standard for conservation laws, the entropy identity for smooth solutions results in the following entropy inequality (in weak form) for weak solutions,

$$(\rho_i s_i)_t + (\rho_i \vec{v}_i^x s_i)_x \leq 0, \quad (\rho_e s_e)_t + (\rho_e \vec{v}_e^x s_e)_x \leq 0$$

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Remark

The above estimates are trivial to generalize to multi-dimensions. The resulting entropy identity is

$$(\rho_i s_i)_t + \mathrm{div}(\rho_i \vec{v}_i s_i) = 0, \quad (\rho_e s_e)_t + \mathrm{div}(\rho_e \vec{v}_e s_e) = 0$$

Energy estimate

Lemma

Assume that there exists constants $\rho_{\alpha,\min}, \rho_{\alpha,\max}, p_{\alpha,\min}$ *such that*

$$\rho_{\alpha,\min} \leq \rho_\alpha \leq \rho_{\alpha,\max}, \quad p_\alpha \geq p_{\alpha,\min},$$

then we have the following estimate,

$$\int_{\mathbb{R}} \|\rho_\alpha\|^2 + \|\rho_\alpha \vec{v}_\alpha\|^2 + \|E_\alpha\|^2 dx \leq C \int_{\mathbb{R}} \rho_\alpha s_\alpha dx + C_1,$$

for some constants C, C_1 *depending on the above parameters.*

Energy estimate

Lemma

Define the electro-magnetic energy as,

$$E_{EM} = \frac{\|\vec{B}\|^2 + \|\phi\|^2}{2} + \frac{\|\vec{E}\|^2 + \|\psi\|^2}{2c^2},$$

then we have the following global estimate,

$$\frac{d}{dt}\int_{\mathbb{R}} E_{EM}dx \leq C_2\left(\int_{\mathbb{R}} E_{EM}dx + \int_{\mathbb{R}}(\rho_i s_i + \rho_e s_e)dx\right) + C_3,$$

where $s, \bar{s}$ are the entropies and C_2, C_3 are constants.

Source term energy

Lemma

Define energy function,

$$E_s = \frac{\rho_i \vec{v}_i^2 + \rho_e \vec{v}_e^2 + \varepsilon_0 \vec{E}^2}{2}.$$

If $\vec{\mathbf{U}}$ *is solutions of*

$$\frac{d\vec{\mathbf{U}}}{dt} = \mathbf{S}(\vec{\mathbf{U}})$$

then,

$$\frac{dE_s}{dt} = 0$$

i.e. energy is conserved. Here $\mathbf{S}$ *is source of two-fluid equation.*

Source Term

Define,

$$\vec{\mathbf{V}} = (\vec{v}_i, \vec{v}_e, \vec{E})^\top$$

then

$$\frac{d\vec{\mathbf{V}}}{dt} = \mathbf{S}_{\vec{\mathbf{V}}}(\vec{\mathbf{V}})$$

Source Term

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then

$$\frac{d\vec{\mathbf{V}}}{dt} = \mathbf{S}_{\vec{\mathbf{V}}}(\vec{\mathbf{V}})$$

with

$$\mathbf{S}(\vec{\mathbf{V}}) = \begin{cases} r_i(\vec{E} + \vec{v}_i \times \vec{B}), \\ r_e(\vec{E} + \vec{v}_e \times \vec{B}), \\ -\frac{1}{\varepsilon_0}\left(r_i \rho_i \vec{v}_i + r_e \rho_e \vec{v}_e\right). \end{cases}$$

Source term eigenvalues

- First three eigenvalues are,

$$0, \qquad \pm i\omega_p$$

where

$$\omega_p^2 = \omega_{pi}^2 + \omega_{pe}^2$$

and

$$\omega_{p\alpha} = \sqrt{\frac{n_\alpha q_\alpha^2}{\epsilon_0 m_\alpha}}, \qquad \alpha \in \{i, e\}$$

- All other eigenvalues are imaginary.
- Source terms are oscillatory in nature.

Non-Dimensional equations

- Number density n_0, Temperature or Pressure ($P_0 = n_0 T_0$).
- Length scale x_0, Magnetic field B_0.

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- Number density n_0, Temperature or Pressure $(P_0 = n_0 T_0)$.
- Length scale x_0, Magnetic field B_0.
- Velocity $V_0 = \sqrt{\frac{P_0}{\rho_0}}$,
- Electric field $E_0 = V_0 B_0$.

Non-Dimensional equations

- Ion

$$
\begin{aligned}
\frac{\partial \rho_i}{\partial t} + \nabla \cdot (\rho_i \vec{v}_i) &= 0, \\
\frac{\partial \rho_i \vec{v}_i}{\partial t} + \nabla \cdot \left(\vec{v}_i \vec{v}_i^{\top} + p_i \mathbf{I} \right) &= \frac{1}{\hat{r}_g} \rho_i (\vec{E} + \vec{v}_i \times \vec{B}), \\
\frac{\partial E_i}{\partial t} + \nabla \cdot ((E_i + p_i) \vec{v}_i) &= \frac{1}{\hat{r}_g} \rho_i (\vec{v}_i \cdot \vec{E})
\end{aligned}
$$

- Electrons

$$
\begin{aligned}
\frac{\partial \rho_e}{\partial t} + \nabla \cdot (\rho_e \vec{v}_e) &= 0, \\
\frac{\partial \rho_e \vec{v}_e}{\partial t} + \nabla \cdot \left(\vec{v}_e \vec{v}_e^{\top} + p_e \mathbf{I} \right) &= -\frac{m_i}{m_e} \frac{1}{\hat{r}_g} \rho_e (\vec{E} + \vec{v}_e \times \vec{B}), \\
\frac{\partial E_e}{\partial t} + \nabla \cdot ((E_e + p_e) \vec{v}_e) &= -\frac{m_i}{m_e} \frac{1}{\hat{r}_g} \rho_e (\vec{v}_e \cdot \vec{E})
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\begin{aligned}
\frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} + \kappa \nabla \psi &= 0, \\
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\frac{\partial \vec{E}}{\partial t} - \hat{c}^2 \nabla \times \vec{B} + \xi \hat{c}^2 \nabla \phi &= -\frac{m_i}{\hat{\lambda}_D^2 \hat{r}_g} \left(r_i \rho_i \vec{v}_i + r_e \rho_e \vec{v}_e \right), \\
\frac{\partial \phi}{\partial t} + \xi \nabla \cdot \vec{E} &= \frac{\xi m_i}{\hat{\lambda}_D^2 \hat{r}_g} (r_i \rho_i + r_e \rho_e).
\end{aligned}
$$

Plasma Paramters

- Larmor radius $r_{g\alpha} = \frac{m_\alpha V_\alpha}{q_\alpha B_0}$ and $\hat{r}_g = r_g / x_0$.
- Debye Length $\lambda_D = \sqrt{\frac{\varepsilon_0 m_\alpha V_\alpha}{n_\alpha q_\alpha}}$ and $\hat{\lambda}_D = \lambda_D / r_g$

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Remark

When Larmor radius $\hat{r}_g \to 0$ TF model approach the MHD limit. SImilarly for $\hat{r}_g \to \infty$ TF model reduce to simple flow equations for ions and electrons. TF equations caputers the intermediate physics of these two limits.

Finite Volume methods

- Key difficulty for FVMs of TF Eqns. is stiff source terms.

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- Key difficulty for FVMs of TF Eqns. is stiff source terms.
- Semi-discrete FVM in 1D,

$$\frac{d\vec{\mathbf{U}}_i}{dt} = -\frac{\left(\vec{\mathbf{F}}_{i+\frac{1}{2}} - \vec{\mathbf{F}}_{i-\frac{1}{2}}\right)}{\Delta x} + S(\vec{\mathbf{U}}_i).$$

Numerical Flux

- Flux has split structure, so we use Euler and Maxwell approximate Riemann solvers.
- Riemann solvers used are Lax-Friedrich, Rusanov, HLLE, HLLE4, HLLE6 and Roe.
- For second order space discretization limiter are used: MinMod, MC, Superbee.

Entropy Conservative flux for TF

Theorem (Tadmor (1987) Math. Comp.)

Consider a one dimensional system of conservation laws with entropy $\mathbf{E}(\vec{\mathbf{U}})$ *and entropy variable* $\vec{\mathbf{V}} = \partial_{\vec{\mathbf{U}}}\mathbf{E}$, *entropy flux* $\mathbf{Q}$, *and entropy potential* $\chi = (\vec{\mathbf{V}}, \vec{\mathbf{F}}) - \mathbf{Q}$. *Let a finite difference scheme with consistent flux satisfying,*

$$([\vec{\mathbf{V}}_{i+\frac{1}{2}}], \vec{\mathbf{F}}^*_{i+\frac{1}{2}}) = [\chi_{i+\frac{1}{2}}]$$

then scheme with this numerical flux satisfy discrete entropy equality and scheme is entropy conservative.

Entropy Conservative flux for TF

- For Linear Maxwell flux part, entropy flux is a simple average.
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- For Euler entropy conservative flux is derived by Roe (Hyp 2006),
- At shocks entropy dissipates.
- Entropy stability using diffusion operator,

$$\vec{\mathbf{F}}_{i+\frac{1}{2}} = \vec{\mathbf{F}}^{*}_{i+\frac{1}{2}} - \frac{1}{2} R_{i+\frac{1}{2}} |\Lambda_{i+\frac{1}{2}}| R^{\top}_{i+\frac{1}{2}} [\vec{\mathbf{V}}_{i+\frac{1}{2}}]$$

- We use Roe and Rusanov diffusion operators.

Explicit Scheme

- Semi-discrete system,

$$\frac{d\vec{\mathbf{U}}^n}{dt} = L(\vec{\mathbf{U}}^n) + S(\vec{\mathbf{U}}^n).$$

- First order Euler,

$$\vec{\mathbf{U}}^{n+1} = \vec{\mathbf{U}}^n + (\Delta t)(L(\vec{\mathbf{U}}^n) + S(\vec{\mathbf{U}}^n)).$$

- TVD-RK[1] for second and third order time stepping.
- Standard RK4 for the fourth order

[1] S. Gottlieb, C. W. Shu, E. Tadmor (2001)

Implicit Scheme

- Semi-discrete system,

$$\frac{d\vec{\mathbf{U}}^n}{dt} = L(\vec{\mathbf{U}}^n) + S(\vec{\mathbf{U}}^{n+1}).$$

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- First order scheme,

$$\vec{\mathbf{U}}^{n+1} = \vec{\mathbf{U}}^n + (\Delta t)(L(\vec{\mathbf{U}}^n) + S(\vec{\mathbf{U}}^{n+1})).$$

So,

$$\begin{aligned}
\rho_\alpha^{n+1} &= \rho_\alpha^n + (\Delta t) L(\vec{\mathbf{U}}^n)_{\rho_\alpha}, \\
\vec{B}^{n+1} &= \vec{B}^n + (\Delta t) L(\vec{\mathbf{U}}^n)_{\vec{B}}, \\
\psi^{n+1} &= \psi^n + (\Delta t) L(\vec{\mathbf{U}}^n)_{\psi}.
\end{aligned}$$

Implicit Scheme

Define

$$\vec{\mathbf{V}} = (\rho_i \vec{v}_i, \rho_e \vec{v}_e, \vec{E})^\top,$$

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$$\vec{\mathbf{V}} = (\rho_i \vec{v}_i, \rho_e \vec{v}_e, \vec{E})^\top,$$

then,

$$\frac{\vec{\mathbf{V}}^{n+1} - \vec{\mathbf{V}}^{n}}{\Delta t} = L(\vec{\mathbf{U}}^{n})_{\vec{\mathbf{V}}} + \mathbf{A}\vec{\mathbf{V}}^{n+1}$$

Implicit Scheme

Define

$$\vec{\mathbf{V}} = (\rho_i \vec{v}_i, \rho_e \vec{v}_e, \vec{E})^\top,$$

then,

$$\frac{\vec{\mathbf{V}}^{n+1} - \vec{\mathbf{V}}^n}{\Delta t} = L(\vec{\mathbf{U}}^n)_{\vec{\mathbf{V}}} + \mathbf{A}\vec{\mathbf{V}}^{n+1}$$

Finally,

$$\vec{\mathbf{V}}^{n+1} = \left(\mathbf{I} - (\Delta t)\mathbf{A}(\vec{\mathbf{U}}^{n+1})\right)^{(-1)} (\vec{\mathbf{V}}^n + (\Delta t) L(\vec{\mathbf{U}}^n)_{\vec{\mathbf{V}}}),$$

- Update E_e, E_i and ϕ at t^{n+1}.

Implicit Scheme

Define

$$\vec{\mathbf{V}} = (\rho_i \vec{v}_i, \rho_e \vec{v}_e, \vec{E})^\top,$$

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- Update E_e, E_i and ϕ at t^{n+1}.
- Second and third order TVD-time stepping with each Euler step replaces by first order update.

Implicit Scheme: Matrix $\mathbf{A}(\vec{\mathbf{U}}^{n+1})$

$$
\begin{bmatrix}
0 & \frac{B^{z,n+1}}{\hat{r}_g} & -\frac{B^{y,n+1}}{\hat{r}_g} & 0 & 0 & 0 & \frac{\rho_i^{n+1}}{\hat{r}_g} & 0 & 0 \\
-\frac{B^{z,n+1}}{\hat{r}_g} & 0 & \frac{B^{x,n+1}}{\hat{r}_g} & 0 & 0 & 0 & 0 & \frac{\rho_i^{n+1}}{\hat{r}_g} & 0 \\
\frac{B^{y,n+1}}{\hat{r}_g} & -\frac{B^{x,n+1}}{\hat{r}_g} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\rho_i^{n+1}}{\hat{r}_g} \\
0 & 0 & 0 & 0 & \frac{B^{z,n+1}}{\hat{r1}_g} & -\frac{B^{y,n+1}}{\hat{r}_{1g}} & \frac{\rho_e^{n+1}}{\hat{r}_{1g}} & 0 & 0 \\
0 & 0 & 0 & -\frac{B^{z,n+1}}{\hat{r}_{1g}} & 0 & \frac{B^{x,n+1}}{\hat{r}_{1g}} & 0 & \frac{\rho_e^{n+1}}{\hat{r}_{1g}} & 0 \\
0 & 0 & 0 & \frac{B^{y,n+1}}{\hat{r}_{1g}} & -\frac{B^{x,n+1}}{\hat{r}_{1g}} & 0 & 0 & 0 & \frac{\rho_e^{n+1}}{\hat{r}_{1g}} \\
\frac{-r_i}{K} & 0 & 0 & \frac{-r_e}{K} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-r_i}{K} & 0 & 0 & \frac{-r_e}{K} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-r_i}{K} & 0 & 0 & \frac{-r_e}{K} & 0 & 0 & 0
\end{bmatrix}
$$

Here $\hat{r}_{1g} = -\frac{m_e}{m_i}\hat{r}_g$ and $K = \frac{\hat{\lambda}_D^2 \hat{r}_g}{m}$

Code Status

- TF FVM Code is developed by using ALSVID MHD code(CMA Oslo).
- Written in C++ with python interface.
- 3D cartesian mesh.
- Parallel version uses MPI.
- MATLAB interface is added for visualization.

Convergence Rates: Forced Solution

- Density $\rho_\alpha = 2 + \sin(2\pi x)$
- $\vec{v}_\alpha = 1.0, p_\alpha = 1.0$
- Mass Ratio $\frac{m_e}{m_i} = \frac{1}{2}$.
- $B^y = \sin(2\pi x), E^z = -\sin(2\pi x)$
- Periodic boundary conditions.
- Added source term for Exact solution to be advection of density profiles.

Convergence Rates: Forced Solution

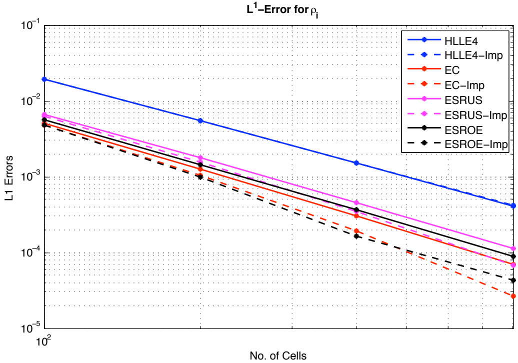

Figure: Error plots for smooth solutions with 2nd order methods

Soliton Propagation:Initial Condition

- Ion density $\rho_i = (1.0 + \exp(-25.0|x - L/3.0|))$ with $L = 12.0$.
- Mass Ratio $\frac{m_e}{m_i} = \frac{1}{25}$.
- Pressure $p_e = 5.0 n_e$ and $p_i = \frac{p_e}{100}$.

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- Mass Ratio $\frac{m_e}{m_i} = \frac{1}{25}$.
- Pressure $p_e = 5.0 n_e$ and $p_i = \frac{p_e}{100}$.
- Reference light speed $\hat{c} = 10.0$, Reference Length $= 100.0$.
- Debye Length 1.0, Larmor Radius =0.01.
- Periodic boundary conditions.

Ref: Baboolal, S. (2001), Hakim A.(2006)

Soliton Propagation:HLLE4 with MinMod

Comparison of time discretizations with 5000 cells

Soliton Propagation: Time Comparison table

Cells	o2exp	o3exp	o4exp	o2imp	o3imp
500	10.65	15.96	21.82	3.15	5.32
1000	21.25	31.52	44.09	12.36	18.58
2000	46.12	69.68	95.29	49.71	75.01
4000	185.05	277.53	396.77	200.46	299.41

Table: Time comparison of the schemes

Soliton Propagation: Comparison of Solvers

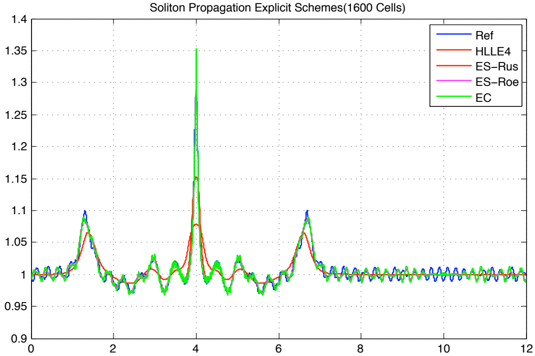
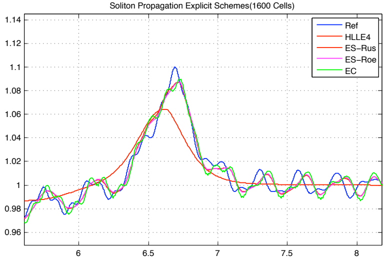


Figure: Solution at $t = 5.0$ for explicit schemes.

Soliton Propagation: Comparison of Solvers



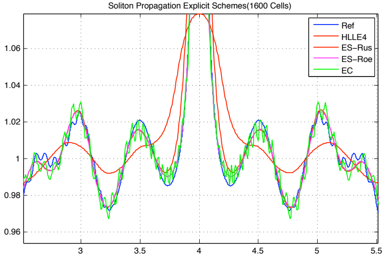


Figure: Solution at $t = 5.0$ for explicit schemes (zoom)

Soliton Propagation: Comparison of Solvers

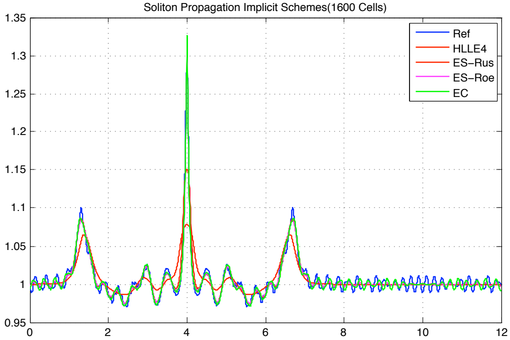


Figure: Solution at $t = 5.0$ for implicit scheme.

Soliton Propagation: Comparison of Solvers

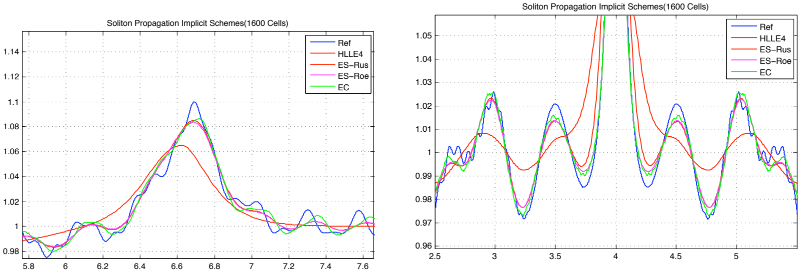


Figure: Solution at $t = 5.0$ for implicit schemes.(zoom)

Brio-Wu:Initial Condition

$$\vec{\mathbf{U}}_{left} = \begin{cases} \rho_i = 1.0 \\ p_i = 5 \times 10^{-5} \\ \rho_e = 1.0 \frac{m_e}{m_i} \\ p_e = 5 \times 10^{-5} \\ B_x = 0.75 \\ B_y = 1.0 \end{cases}$$

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$$\vec{\mathbf{U}}_{left} = \begin{cases} \rho_i = 1.0 \\ p_i = 5 \times 10^{-5} \\ \rho_e = 1.0 \frac{m_e}{m_i} \\ p_e = 5 \times 10^{-5} \\ B_x = 0.75 \\ B_y = 1.0 \end{cases} \qquad \vec{\mathbf{U}}_{Right} = \begin{cases} \rho_i = 0.125 \\ p_i = 5 \times 10^{-6} \\ \rho_e = 0.125 \frac{m_e}{m_i} \\ p_e = 5 \times 10^{-6} \\ B_x = 0.75 \\ B_y = -1.0 \end{cases}$$

Brio-Wu:Initial Condition

$$\vec{\mathbf{U}}_{left} = \begin{cases} \rho_i = 1.0 \\ p_i = 5 \times 10^{-5} \\ \rho_e = 1.0 \frac{m_e}{m_i} \\ p_e = 5 \times 10^{-5} \\ B_x = 0.75 \\ B_y = 1.0 \end{cases} \qquad \vec{\mathbf{U}}_{Right} = \begin{cases} \rho_i = 0.125 \\ p_i = 5 \times 10^{-6} \\ \rho_e = 0.125 \frac{m_e}{m_i} \\ p_e = 5 \times 10^{-6} \\ B_x = 0.75 \\ B_y = -1.0 \end{cases}$$

- Mass Ratio $\frac{m_e}{m_i} = \frac{1}{1832.6}$
- Pressure $p_0 = 10^{-4}$
- Light speed $\hat{c} = 100.$
- Debye Length 0.01

Ref: Shumlak, U.(2003), Loverich J.(2005), Hakim A.(2006)

Brio-Wu:Solution with $\hat{r}_g = 50.0$

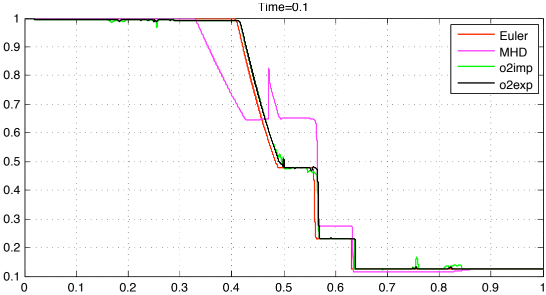

Figure: Comparison of o2exp and o2imp with 20000 cells

Brio-Wu:Solution with $\hat{r}_g = 50.0$

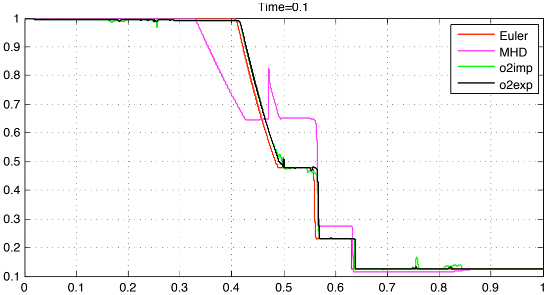

Figure: Comparison of o2exp and o2imp with 20000 cells

Time: o2exp 16073.35 sec vs o2imp 17482.41 sec

Brio-Wu:Solution with $\hat{r}_g = 0.005$

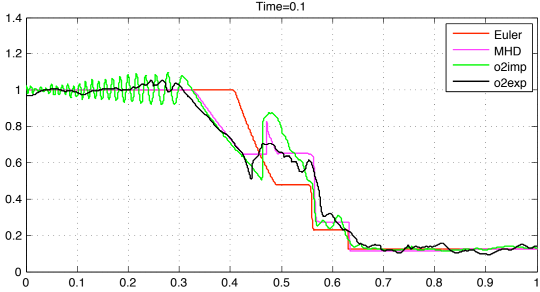

Figure: Comparison of o2exp and o2imp with 20000 cells

Brio-Wu:Solution with $\hat{r}_g = 0.005$

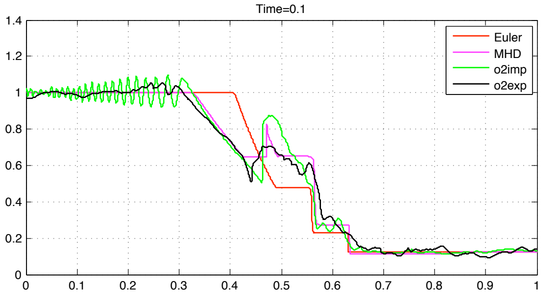

Figure: Comparison of o2exp and o2imp with 20000 cells

Time: o2exp 27377.6 sec vs o2imp 13613.042 sec

Brio-Wu:Solution with $\hat{r}_g = 10.0$

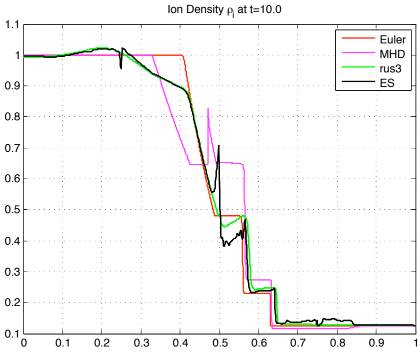

Figure: Comparison of Rus3 with ES-Rus with 1000 cells

Brio-Wu:Solution with $\hat{r}_g = 0.01$

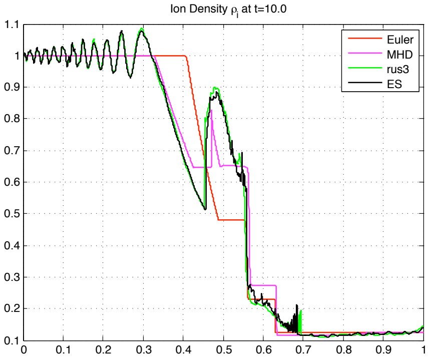

Figure: Comparison of Rus3 with ES-Rus with 10000 cells

Magnetic Reconnection:Initial Conditions

- Domain $D = [-L_x/2, L_x/2] \times [-L_y/2, L_y/2]$ with $Lx = 8\pi$ and $L_y = 4\pi$.
- Number densities $n = n_e = n_i = (\frac{1}{5} + sech^2(\frac{y}{\lambda}))$, with $\lambda = 0.5$.
- $\vec{B} = B_0 \tanh\left(\frac{y}{\lambda}\right) e_x$, with $B_0 = 0.1$
- $\vec{J_e} = -\frac{B_0}{\lambda} sech^2\left(\frac{y}{\lambda}\right)$
- $p_e = \frac{B_0}{12} n(y)$ and $p_i = 5.0 p_e$.
- Magnetic perturbation, $\delta\vec{B} = \vec{e_z} \times \nabla\chi$ with $\chi = \chi_0 \cos(\frac{2\pi x}{L_x}) \cos\frac{\pi y}{L_y}$ and $\chi_0 = \frac{B_0}{10.0}$

Magnetic Reconnection:Non-Dimensional Variables

- $B_0 = 0.1, p_0 = 0.01, V_0 = 0.1.$
- Debye Length 0.1
- Larmor Radius 1.0, Light speed $\hat{c} = 10.0$.
- Boundary condition: Periodic in x-direction. Conductive in y-direction.

Magnetic Reconnection: $|J_z|$ at $t = 25.0$

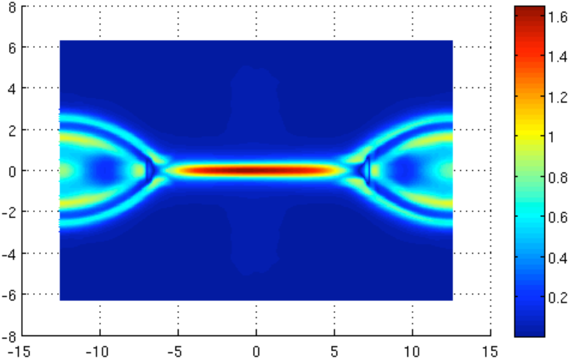

Figure: O3exp on 800×400 mesh

Magnetic Reconnection

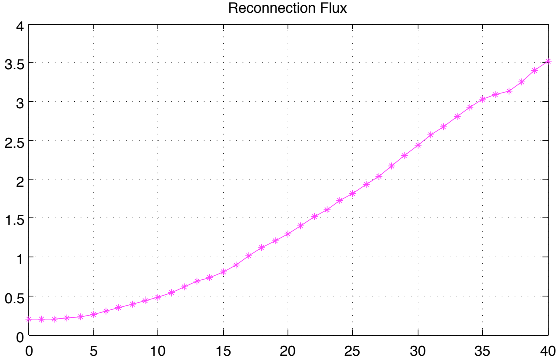

Figure: Reconnection Flux vs time

Conclusions

- Presented various entropy and energy estimates for TF eqns..
- Time step constraint imposed by stiff source is overcome by implicit scheme.
- For non-stiff problems both schemes are comparable.
- Developed entropy stable numerical scheme for TF.

Future Work

- Higher order entropy stable discretization.
- Further analysis of the time step constraint imposed by stiff source.
- Implicit time stepping for Maxwell part.
- Compute Magnetic Reconnection.

Future Work

THANK YOU