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Adaptive Stochastic Galerkin Methods

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Outline

A model problem

$D \subset \mathbb{R}^d$ Lipschitz, for $\omega \in \Omega$,

$$-\nabla \cdot (a(\omega, x)\nabla u(\omega, x)) = f(x) \qquad x \in D\,,$$
$$u(\omega, x) = 0 \qquad x \in \partial D\,.$$

Weak formulation: Find $u(\omega, x) \in H_0^1(D)$ such that

$$\int_D a(\omega, x)\nabla u(\omega, x) \cdot \nabla v(x)\mathrm{d}x = \int_D f(x)v(x)\mathrm{d}x$$

for all $v \in H_0^1(D)$.

Assume $a(\omega, x)$ is uniformly bounded,

$$0 < a_- \leq a(\omega, x) \leq a_+ < \infty \qquad \forall x \in D\,, \forall \omega \in \Omega\,.$$

Series expansion

Let $\bar{a}(x)$ be some approximation of $a(\omega, x)$, *e.g.* $\bar{a}(x) := (a_- + a_+)/2$.

Let $(\psi_m, \psi_m^*)_{m\in\mathbb{N}}$ be a biorthogonal basis of $L^2(D)$. For $m \in \mathbb{N}$,

$$Y_m(\omega) := \int_D (a(\omega, x) - \bar{a}(x))\psi_m^*(x)\mathrm{d}x \in [-1, 1] .$$

By Hölder's inequality, Y_m is bounded for all $m \in \mathbb{N}$. Without loss of generality, ψ_m, ψ_m^* are scaled such that $|Y_m| \leq 1$.

Then

$$a(\omega, x) = \bar{a}(x) + \sum_{m=1}^{\infty} Y_m(\omega)\psi_m(x) ,$$

with convergence in $L^2(D)$ for all $\omega \in \Omega$.

Under stronger assumptions, *e.g.* $\sum_{m=1}^{\infty} \|\psi_m\|_{L^\infty(D)} < \infty$, also convergence in $L^\infty(D)$, uniformly in $\omega \in \Omega$.

Transformation to a product parameter domain

The coefficients $Y_m(\omega)$ of $a(\omega, x) - \bar{a}(x)$ constitute a map

$$Y : \Omega \to \Gamma := [-1, 1]^\infty , \quad \omega \mapsto (Y_m(\omega))_{m \in \mathbb{N}} .$$

Define the parametric diffusion coefficient

$$a(y, x) := \bar{a}(x) + \sum_{m=1}^{\infty} y_m \psi_m(x) , \quad y = (y_m)_{m \in \mathbb{N}} \in \Gamma .$$

Then $a(Y(\omega), x) = a(\omega, x)$ for all $\omega \in \Omega$.

For all $y \in \Gamma$, let $u(y) \in H_0^1(D)$ solve

$$\int_D a(y, x) \nabla u(y, x) \cdot \nabla v(x) \mathrm{d}x = \int_D f(x) v(x) \mathrm{d}x \quad \forall v \in H_0^1(D) .$$

Then $u(Y(\omega), x) = u(\omega, x)$ for all $\omega \in \Omega$.

Weak formulation on the parameter domain

Let π_m be a probability measure on $[-1,1]$ for all $m \in \mathbb{N}$, and

$$\pi := \bigotimes_{m=1}^{\infty} \pi_m \quad \text{on} \quad (\Gamma, \mathscr{B}(\Gamma)) \,.$$

If $Y \sim \pi$, then $(Y_m)_m$ are independent, and $Y_m \sim \pi_m$.

Weak formulation: Find $u \in L^2_\pi(\Gamma; H^1_0(D))$ s.t. $\forall v \in L^2_\pi(\Gamma; H^1_0(D))$,

$$\int_\Gamma \int_D a(y,x) \nabla u(y,x) \cdot \nabla v(y,x) \mathrm{d}x \mathrm{d}\pi(y) = \int_\Gamma \int_D f(x) v(y,x) \mathrm{d}x \mathrm{d}\pi(y) \,.$$

Continuous and coercive bilinear form on $L^2_\pi(\Gamma; H^1_0(D))$.

Product structure: $L^2_\pi(\Gamma; H^1_0(D)) \cong L^2_\pi(\Gamma) \otimes H^1_0(D)$.

- deterministic space $H^1_0(D)$,
- stochastic space $L^2_\pi(\Gamma)$.

Orthonormal polynomials

Orthonormal polynomials w.r.t. π_m satisfy a recursion $P_0^{(m)} = 1$,

$$\beta_{n+1}^{(m)} P_{n+1}^{(m)}(\xi) = (\xi - \alpha_n^{(m)}) P_n^{(m)}(\xi) - \beta_n^{(m)} P_{n-1}^{(m)}(\xi) .$$

Tensor product orthonormal polynomials

$$P_\mu := \bigotimes_{m=1}^{\infty} P_{\mu_m}^{(m)} , \quad \mu \in \Lambda := \left\{ \mu \in \mathbb{N}_0^{\mathbb{N}} \; ; \; \# \operatorname{supp} \mu < \infty \right\} ,$$

i.e. $P_\mu(y) = \prod_{m=1}^{\infty} P_{\mu_m}^{(m)}(y_m) = \prod_{m \in \operatorname{supp} \mu} P_{\mu_m}^{(m)}(y_m)$ for $y \in \Gamma$.

Theorem

$(P_\mu)_{\mu \in \Lambda}$ *is an orthonormal basis of* $L^2_\pi(\Gamma)$.

Infinite system of equations

Expand the solution u w.r.t. the basis $(P_\mu)_{\mu\in\Lambda}$ of $L^2_\pi(\Gamma)$,

$$u(y,x) = \sum_{\mu\in\Lambda} u_\mu(x) P_\mu(y)\,, \quad (u_\mu)_{\mu\in\Lambda} \in \ell^2\big(\Lambda; H^1_0(D)\big)\ .$$

Define $\quad a_{\nu\mu}(x) := \int_\Gamma a(y,x) P_\mu(y) P_\nu(y) \mathrm{d}\pi(y)\,, \quad \mu,\nu\in\Lambda$,

and $\quad f_\nu(x) := f(x) \int_\Gamma P_\nu(y) \mathrm{d}\pi(y)\,, \quad \nu\in\Lambda$.

Then for all $\nu\in\Lambda$, using test functions $v(x)P_\nu(y)$,

$$\sum_{\mu\in\Lambda} \int_D a_{\nu\mu}(x) \nabla u_\mu(x)\cdot\nabla v(x) \mathrm{d}x = \int_D f_\nu(x) v(x) \mathrm{d}x \quad \forall v\in H^1_0(D)\ .$$

Infinite system of deterministic equations for the coefficients of u, equivalent to the weak formulation by Parseval's identity.

Infinite system of equations

Recall

- $a_{\nu\mu}(x) = \int_\Gamma a(y,x) P_\mu(y) P_\nu(y) \mathrm{d}\pi(y)$, $\mu, \nu \in \Lambda$,
- $\xi P_n^{(m)}(\xi) = \beta_{n+1}^{(m)} P_{n+1}^{(m)}(\xi) + \alpha_n^{(m)} P_n^{(m)}(\xi) + \beta_n^{(m)} P_{n-1}^{(m)}(\xi)$,
- $a(y,x) = \bar{a}(x) + \sum_{m=1}^{\infty} y_m \psi_m(x)$.

Therefore, $a_{\nu\mu} = 0$ except for

$$a_{\mu\mu}(x) = \bar{a}(x) + \sum_{m=1}^{\infty} \alpha_{\mu_m}^{(m)} \psi_m(x) ,$$

$$a_{\mu,\mu+\epsilon_m}(x) = a_{\mu+\epsilon_m,\mu}(x) = \beta_{\mu_m+1}^{(m)} \psi_m(x) , \quad m \in \mathbb{N} .$$

In particular, no integrals over Γ need to be computed.

Galerkin projection

Define a nested sequence of finite element spaces

$$\{0\} = V_0 \subset V_1 \subset \ldots \subset V_\ell \subset V_{\ell+1} \subset \ldots \subset H_0^1(D) .$$

For each $\mu \in \Lambda$, select a space $V_{\ell(\mu)}$, and define

$$\mathscr{V}_\ell := \left\{ (v_\mu)_{\mu \in \Lambda} \, ; \, v_\mu \in V_{\ell(\mu)} \; \forall \mu \in \Lambda \right\} \subset \ell^2\big(\Lambda; H_0^1(D)\big) .$$

Assume $\#\{\mu \in \Lambda \, ; \, \ell(\mu) \neq 0\} < \infty$, then $\dim \mathscr{V}_\ell < \infty$.

Galerkin projection: replace $\ell^2(\Lambda; H_0^1(D))$ by $\mathscr{V}_\ell$ in the infinite system of equations for $(u_\mu)_{\mu \in \Lambda}$.

- finite system of equations since $\mathscr{V}_\ell$ is finite dimensional.
- well-posed by Lax–Milgram since the bilinear form is coercive.
- quasi-optimal approximation in $\ell^2(\Lambda; H_0^1(D)) \cong L^2_\pi(\Gamma; H_0^1(D))$.

Operator matrix perspective

Semidiscrete setting $\mathscr{V}_N := \ell^2(\Lambda_N; H^1_0(D))$ for a $\Lambda_N \subset \Lambda$, $\#\Lambda_N = N$.

Define the operator matrix $\boldsymbol{A}_N \in \mathscr{L}(H^1_0(D), H^{-1}(D))^{N\times N}$,

$$\boldsymbol{A}_N := (A_{\nu\mu})_{\nu,\mu\in\Lambda_N}\,, \quad A_{\nu\mu}v := -\nabla\cdot(a_{\nu\mu}\nabla v)\,,$$

and the vector $\boldsymbol{f}_N := (f_\nu)_{\nu\in\Lambda_N} \in H^{-1}(D)^N$.

The Galerkin projection $\boldsymbol{u}_N \in H^1_0(D)^N$ is the solution of

$$\boldsymbol{A}_N\boldsymbol{u}_N = \boldsymbol{f}_N\,.$$

Iterative solution, *e.g.* by PCG, with preconditioner $\overline{\boldsymbol{A}}_N := \operatorname{diag}(\bar{A})$, $\bar{A}v := -\nabla\cdot(\bar{a}\nabla v)$ consists of applications of $A_{\nu\mu}$ and $\bar{A}^{-1}$.

The matrix $\boldsymbol{A}_N$ is sparse,

$$\operatorname{nnz}(\boldsymbol{A}_N) \leq (1 + 2\bar{\lambda}(\Lambda_N))N$$

for the average index length $\bar{\lambda}(\Lambda_N) := \frac{1}{N}\sum_{\mu\in\Lambda_N} \#\operatorname{supp}\mu$.

Adaptive Method

Iterative adaptive method $\boldsymbol{u}_N^{(0)} := \boldsymbol{0}$, $\Lambda_N^{(0)} := \emptyset$

1. $\boldsymbol{r}_N^{(k)} := \mathrm{Residual}(\boldsymbol{u}_N^{(k)})$ approximate $\boldsymbol{f} - \boldsymbol{A}\boldsymbol{u}_N^{(k)}$
2. $\Lambda_N^{(k+1)} := \mathrm{Refine}(\Lambda_N^{(k)}, \boldsymbol{r}_N^{(k)})$ refine the discretization
3. $\boldsymbol{u}_N^{(k+1)} := \mathrm{Galerkin}(\Lambda_N^{(k+1)})$ Galerkin sol. in $\ell^2(\Lambda_N^{(k+1)}; H_0^1(D))$

Lemma

If $\left\| (\boldsymbol{f} - \boldsymbol{A}\boldsymbol{u}_N^{(k)})\big|_{\Lambda_N^{(k+1)}} \right\|_{\ell^2(\Lambda;H^{-1})} \geq \vartheta \left\| \boldsymbol{f} - \boldsymbol{A}\boldsymbol{u}_N^{(k)} \right\|_{\ell^2(\Lambda;H^{-1})}$, *then*

$$\left\| \boldsymbol{u} - \boldsymbol{u}_N^{(k+1)} \right\|_{\boldsymbol{A}} \leq \sqrt{1 - \kappa(\boldsymbol{A})^{-1}\vartheta^2} \left\| \boldsymbol{u} - \boldsymbol{u}_N^{(k)} \right\|_{\boldsymbol{A}} .$$

In particular, the iteration converges in the energy norm.

Approximation of the residual

Goal

For $w = \sum_{\mu \in \Lambda_N} w_\mu P_\mu$, approximate the residual

$$r(w) := f + \nabla \cdot (\bar{a} \nabla w) + \sum_{m=1}^{\infty} \nabla \cdot (\psi_m \nabla w) .$$

Partition

$$\Lambda_N = \Lambda_N^{[1]} \sqcup \Lambda_N^{[2]} \sqcup \cdots \sqcup \Lambda_N^{[J]} ,$$

where $\Lambda_N^{[1J]}$ contains $\mu \in \Lambda_N$ with the largestsmallest $|w_\mu|_{H_0^1(D)}$.

For $w^{[j]} := w|_{\Lambda_N^{[j]}}$ and $M_1 \geq M_2 \geq \cdots \geq M_J \geq 0$,

$$r(w) \approx r_N(w) := f + \nabla \cdot (\bar{a} \nabla w) + \sum_{j=1}^{J} \sum_{m=1}^{M_j} \nabla \cdot (\psi_m \nabla w^{[j]}) .$$

Computational example

Model problem

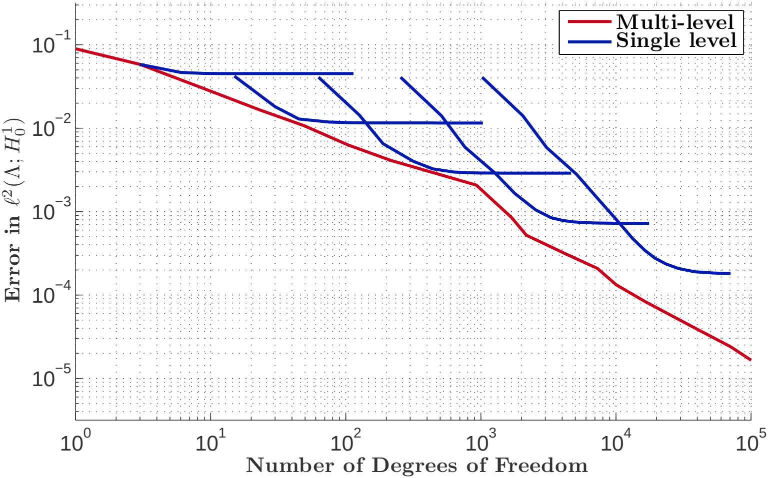
- $D = (0, 1)$
- $f(x) = x$
- $\bar{a}(x) = 1$, *i.e.* $\bar{A} = -\Delta$
- $\psi_m(x) = cm^{-4} \sin(m\pi x)$, scaled such that $c \sum_{m=1}^{\infty} m^{-4} = \frac{5}{6}$
- π_m is the uniform distribution on $[-1, 1]$, *i.e.* $\mathrm{d}\pi_m(y_m) = \frac{1}{2}\mathrm{d}y_m$
- $P_n^{(m)}$ are Legendre polynomials

Linear finite element discretization,

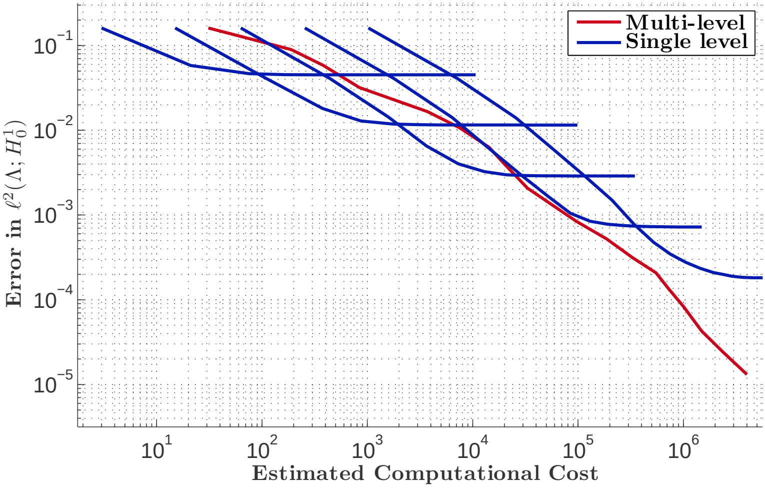
Single level: fixed finite element space $V_\ell \subset H_0^1(D)$ for all $\mu \in \Lambda_N$ and all iterations k

Multi-level: different V_ℓ for each $\mu \in \Lambda_N$ and $k \in \mathbb{N}$, determined a posteriori in the refinement step

Computational example



Computational example



Summary

- Tensor products of orthonormal polynomials form an orthonormal basis $(P_\mu)_{\mu\in\Lambda}$ of $L^2_\pi(\Gamma)$ for the infinite product domain $\Gamma = [-1,1]^\infty$ and infinite product measure $\pi = \bigotimes_{m=1}^\infty \pi_m$.
- Expanding the solution u w.r.t. this basis, a stochastic boundary value problem is recast as an infinite system of deterministic equations.
- Galerkin projections can be computed by standard iterative methods applied to a sparse operator matrix.
- Adaptive wavelet techniques applied to the basis $(P_\mu)_{\mu\in\Lambda}$ can be used to determine appropriate sets Λ_N of active indices and finite element spaces $V_{\ell(\mu)}$, $\mu \in \Lambda_N$, during the solution process.

Thank you for your attention!