Finite difference schemes for nonconservative hyperbolic systems

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- The hyperbolic conservation law

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0 \tag{1}$$

($\mathbf{u} : (a, b) \times [0, \infty)| \to \mathbb{R}^m$) has non-smooth solutions, and so must be interpreted in a weak, distributional manner:

$$\int_0^\infty \int_a^b \mathbf{u}\psi_t + \mathbf{f}(\mathbf{u})\psi_x dxdt + \int_a^b \mathbf{u}(x, 0)\psi(x, 0)dx = 0$$

for all $\psi \in C_0^\infty((a, b) \times [0, \infty))$. To obtain uniqueness, *entropy conditions* must be added.

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$$(\text{mpy. by } \eta'(\mathbf{u})^\top) \qquad \Rightarrow \qquad \eta(\mathbf{u})_t + q(\mathbf{u})_x = 0.$$

Entropy should be *dissipated* at shocks, giving the **entropy condition**

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- Have existence, uniqueness globally for scalar conservation laws and locally for systems.

Nonconservative systems – the problem of multiplication

- Can write (1) as $\mathbf{u}_t + \mathbf{f}'(\mathbf{u})\mathbf{u}_x = 0$. More generally:

$$\mathbf{w}_t + \mathbf{g}(\mathbf{w})\mathbf{w}_x = 0 \qquad (2)$$

  for some $\mathbf{g} \in C(\mathbb{R}^n; \mathbb{R}^{n\times n})$.
- $\mathbf{g}(\mathbf{w})\mathbf{w}_x$ for $\mathbf{w} \in \mathrm{BV}((a,b) \times \mathbb{R}_+)$ is a *nonconservative product*.

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- (2) is *not* defined when $\mathbf{w}$ is discontinuous – we cannot throw derivatives onto a test function ψ.
- **Example:** If H is the Heaviside function

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

  then $\frac{dH}{dx} = \delta_0$, the Dirac measure, but

$$H\frac{dH}{dx} = H\delta_0$$

  is undefined at 0.

DLM theory of nonconservative products

Fix $t \in \mathbb{R}_+$ and consider $\mathbf{w}$ as a function in $\mathrm{BV}((a,b);\mathbb{R}^n)$.

- Theory of Dal Maso, LeFloch and Murat [3]: Define $\mathbf{g}(\mathbf{w})\frac{d\mathbf{w}}{dx}$ as a measure μ:
  - If $\mathbf{w}$ is continuous in $B \subset (a,b)$ then

$$\mu(B) := \int_B \mathbf{g}(\mathbf{w}) \left(\frac{d\mathbf{w}}{dx}\right)$$

($\frac{d\mathbf{w}}{dx}$ is a Borel measure and $g(\mathbf{w})$ is continuous, so this is well-defined).

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$$\mu(\{x\}) := \int_0^1 \mathbf{g}\left(\phi(s;\mathbf{w}(x^-),\mathbf{w}(x^+))\right)\frac{\partial\phi}{\partial s}(s;\mathbf{w}(x^-),\mathbf{w}(x^+))ds$$

  where $\phi : [0,1]\times\mathbb{R}^n\times\mathbb{R}^n$ is a *family of paths:* for all $\mathbf{w_L},\mathbf{w_R},\mathbf{w}\in\mathbb{R}^n$ we have

$$\phi(0;\mathbf{w_L},\mathbf{w_R}) = \mathbf{w_L}, \qquad \phi(1,\mathbf{w_L},\mathbf{w_R}) = v_{\mathbf{R}}, \qquad \phi(s,\mathbf{w},\mathbf{w}) \equiv \mathbf{w} \quad \forall\ s\in[0,1].$$

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- We denote $\left[\mathbf{g}(\mathbf{w})\frac{d\mathbf{w}}{dx}\right]_\phi = \mu$.
- **Theorem:** $\left[\mathbf{g}(\mathbf{w})\frac{d\mathbf{w}}{dx}\right]_\phi$ is a bounded Borel measure.

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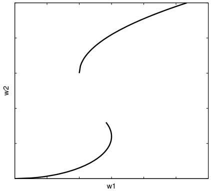

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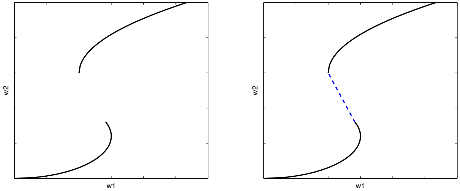

Weak solutions

Recall that a weak solution of (1) satisfies

$$\int_0^\infty \int_a^b \mathbf{u}\psi_t + \mathbf{f}(\mathbf{u})\psi_x dxdt + \int_a^b \mathbf{u}(x,0)\psi(x,0)dx = 0$$

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Definition

A function $\mathbf{w} \in L^\infty\big([0,\infty); \mathrm{BV}((a,b);\mathbb{R}^n)\big)$ is a weak solution of (2) if

$$\int_0^\infty \int_a^b \mathbf{w}\psi_t\, dx - \langle [\mathbf{g}(\mathbf{w}(\cdot,t))\mathbf{w}_x(\cdot,t)]_\phi, \psi(\cdot,t)\rangle dt + \int_a^b \mathbf{w}(x,0)\psi(x,0)dx = 0$$

for all $\psi \in C_0^\infty((a,b)\times[0,\infty))$.

Here, $\langle\cdot,\cdot\rangle$ denotes the pairing

$$\langle [\mathbf{g}(\mathbf{w}(\cdot,t))\mathbf{w}_x(\cdot,t)]_\phi, \psi(\cdot,t)\rangle = \int_{(a,b)} \psi(x,t)\, [\mathbf{g}(\mathbf{w}(\cdot,t))\mathbf{w}_x(\cdot,t)]_\phi\,(x),$$

where the integral is with respect to the measure $[\mathbf{g}(\mathbf{w}(\cdot,t))\mathbf{w}_x(\cdot,t)]_\phi$.

- We discretize the domain into intervals $I_j = [x_{j-1/2}, x_{j+1/2}]$, with $x_{j+1/2} - x_{j-1/2} \equiv \Delta x$. We solve for

$$\mathbf{u}_j^n \approx \frac{1}{\Delta x} \int_{I_j} \mathbf{u}(x, t^n).$$

- A finite volume scheme for (1) is *conservative* if it is of the form

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \frac{\Delta t}{\Delta x} \left(\mathbf{F}_{j+1/2} - \mathbf{F}_{j-1/2} \right),$$

where $\mathbf{F}_{j+1/2} = \mathbf{F}(\mathbf{u}_j^n, \mathbf{u}_{j+1}^n)$ satisfies $\mathbf{F}(\mathbf{u}, \mathbf{u}) = \mathbf{f}(\mathbf{u})$ for all $u \in \mathbb{R}^n$.

Conservative and path conservative schemes

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- A scheme for (2) is *path conservative with respect to* ϕ if it is of the form
$$\mathbf{w}_j^{n+1} = \mathbf{w}_j^n - \frac{\Delta t}{\Delta x} \left(\mathbf{D}_{j+1/2}^- + \mathbf{D}_{j-1/2}^+ \right)$$
for some $\mathbf{D}_{j+1/2}^\pm = \mathbf{D}^\pm(\mathbf{w}_j^n, \mathbf{w}_{j+1}^n)$ which satisfies
  - $\mathbf{D}^\pm(\mathbf{w}, \mathbf{w}) = 0$ for all $\mathbf{w} \in \mathbb{R}^n$
  - $\mathbf{D}_{j+1/2}^- + D_{j+1/2}^+ = \int_0^1 \mathbf{g}(\phi(s; \mathbf{w}_j, \mathbf{w}_{j+1})) \frac{\partial \phi}{\partial s}(s; \mathbf{w}_j, \mathbf{w}_{j+1}) ds.$
- These two definitions are equivalent if $\mathbf{g}(\mathbf{u}) = \mathbf{f}'(\mathbf{u})$ for some $\mathbf{f}$.

Deficiencies of path conservative schemes

- Pointwise convergence of numerical approximations $\mathbf{w}^{\Delta}$ to some $\mathbf{w}$ does *not* imply that $\mathbf{w}$ solves (2) (no Lax-Wendroff theorem).

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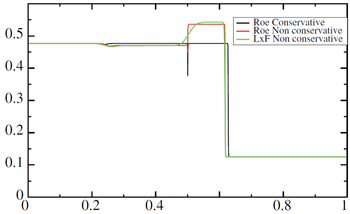
- Recent studies [1, 2] have shown that path conservative schemes can indeed converge to *wrong* solutions, with incorrect shock speeds, intermediate states, etc.



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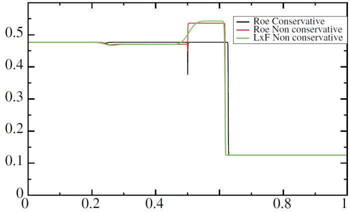

Theorem (Castro et. al. [2])

Let $\mathbf{w}^{\Delta x}$ be computed by a path conservative scheme and assume that $\|\mathbf{w}^{\Delta x}(\cdot, t)\|_{\mathrm{BV}} \leq C$ uniformly in time. If $\mathbf{w}^{\Delta x} \to \mathbf{w}$ pointwise a.e. as $\Delta x \to 0$, then

(i) *There is a bounded measure $\lambda : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^n$ such that*

$$\mathbf{w_t} + [\mathbf{g}(\mathbf{w})\mathbf{w_x}]_\phi = \lambda.$$

(ii) *If the ϕ-graph completions of $\mathbf{w}^{\Delta x}$ converge uniformly to that of $\mathbf{w}$, then*

$$\mathbf{w_t} + [\mathbf{g}(\mathbf{w})\mathbf{w_x}]_\phi = 0.$$

- Consider the model system

$$\begin{aligned} v_t - u_x &= 0 \\ u_t + p_x &= 0 \\ E_t + (pu)_x &= 0, \end{aligned} \tag{3}$$

with

$$E = e + \frac{u^2}{2}, \qquad e = \frac{pv}{\gamma - 1}.$$

Here, v is specific volume, u velocity, p gas pressure, E total energy and e internal energy.

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- Abgrall and Karni [1] consider the equivalent, *nonconservative* system

$$\begin{aligned} v_t - u_x &= 0 \\ u_t + p_x &= 0 \\ e_t + pe_x &= 0. \end{aligned} \tag{4}$$

No path conservative schemes for (4) were found that converge to the entropy solution of (3) (the "correct" solution).

- Consider instead the following parabolic, regularized form of (3):

$$\begin{aligned} v_t - u_x &= \varepsilon v_{xx} \\ u_t + p_x &= \varepsilon u_{xx} \\ E_t + (pu)_x &= \varepsilon E_{xx}. \end{aligned} \tag{5}$$

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- We will discretize (5) and (6) using **entropy conservative** schemes for the left-hand sides and **central differences** for the right-hand sides.

- Entropy stable scheme for conservative system (5):

$$
\begin{aligned}
\frac{d}{dt}v_j - \frac{u_{j+1}-u_{j-1}}{2\Delta x} &= \varepsilon \frac{v_{j+1}-2v_j+v_{j-1}}{\Delta x^2} \\
\frac{d}{dt}u_j + \frac{p_{j+1}-p_{j-1}}{2\Delta x} &= \varepsilon \frac{u_{j+1}-2u_j+u_{j-1}}{\Delta x^2} \\
\frac{d}{dt}E_j + p_j\frac{u_{j+1}-u_{j-1}}{2\Delta x} + u_j\frac{p_{j+1}-p_{j-1}}{2\Delta x} &= \varepsilon \frac{E_{j+1}-2E_j+E_{j-1}}{\Delta x^2}
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- Entropy stable scheme for nonconservative system (6):

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\end{aligned}$$

- To avoid having to use very small Δx in order to resolve the viscous profile, we set

$$\varepsilon = \frac{c^n}{2}\Delta x,$$

where $c^n := \max_j |c_j^n|$ and c_j^n are the eigenvalues of $\mathbf{f}'(\mathbf{u}(x_j, t^n))$.

Numerical experiment

Let

$$(v_0(x), u_0(x), p_0(x)) = \begin{cases} (8, 0, 0.1) & \text{if} \quad x < 0.5 \\ (2.0984, 2.3047, 1) & \text{if} \quad x > 0.5. \end{cases}$$

The exact solution should be a single right-going shock.

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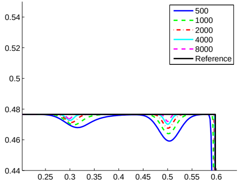

(a) Conservative approximation.

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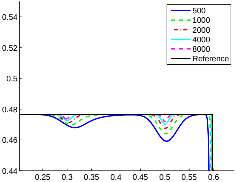
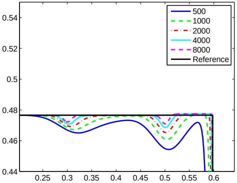


(a) Conservative approximation.

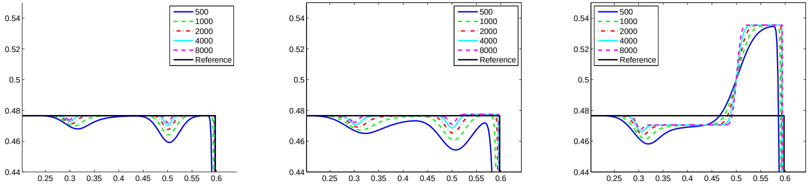


(b) Nonconservative approximation.

Let

$$(v_0(x), u_0(x), p_0(x)) = \begin{cases} (8, 0, 0.1) & \text{if} \quad x < 0.5 \\ (2.0984, 2.3047, 1) & \text{if} \quad x > 0.5. \end{cases}$$

The exact solution should be a single right-going shock.



(a) Conservative approximation. (b) Nonconservative approximation. (c) Nonconservative approximation without u_x^2 term.

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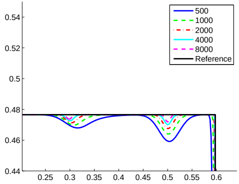

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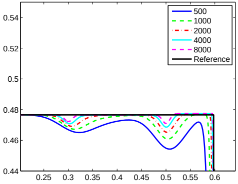
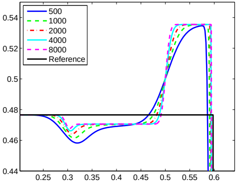
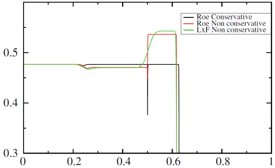


(b) Nonconservative approximation.



(c) Nonconservative approximation without u_x^2 term.



Consider the Isothermal Euler equations [4]

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + \rho)_x = 0, \end{cases} \qquad \begin{cases} \rho_t + (\rho u)_x = 0 \\ u_t + \left(\frac{u^2}{2} + \log\rho\right)_x = 0. \end{cases} \tag{7}$$

We regularize these as

$$\begin{cases} \rho_t + (\rho u)_x = \varepsilon \rho_{xx} \\ (\rho u)_t + (\rho u^2 + \rho)_x = \varepsilon (\rho u)_{xx}, \end{cases} \qquad \begin{cases} \rho_t + (\rho u)_x = \varepsilon \rho_{xx} \\ u_t + \left(\frac{u^2}{2} + \log\rho\right)_x = \varepsilon u_{xx} + 2\varepsilon(\log\rho)_x u_x. \end{cases} \tag{8}$$

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We discretize the regularized nonconservative system as before with **entropy conservative** schemes for the left-hand side and **central differences** for the right-hand side, obtaining

$$\frac{d}{dt}\rho_j + \frac{\rho_{j+1}u_{j+1} - \rho_{j-1}u_{j-1}}{2\Delta x} = \varepsilon \frac{\rho_{j+1} - 2\rho_j + \rho_{j-1}}{\Delta x^2}$$

$$\frac{d}{dt}(\rho_j u_j) + \frac{u_{j+1}^2 - u_{j-1}^2}{4\Delta x} + \frac{\log \rho_{j+1} - \log \rho_{j-1}}{2\Delta x} = \varepsilon \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} + 2\varepsilon \left(\frac{\log \rho_{j+1} - \log \rho_{j-1}}{2\Delta x}\right)\left(\frac{u_{j+1} - u_{j-1}}{2\Delta x}\right)$$

Numerical experiment

We consider the following numerical experiment, taken from [4]:

$$(\rho_0(x), u_0(x)) = \begin{cases} (0.4, 1) & \text{if} \quad x < 0.5 \\ (0.1, 0) & \text{if} \quad x > 0.5. \end{cases}$$

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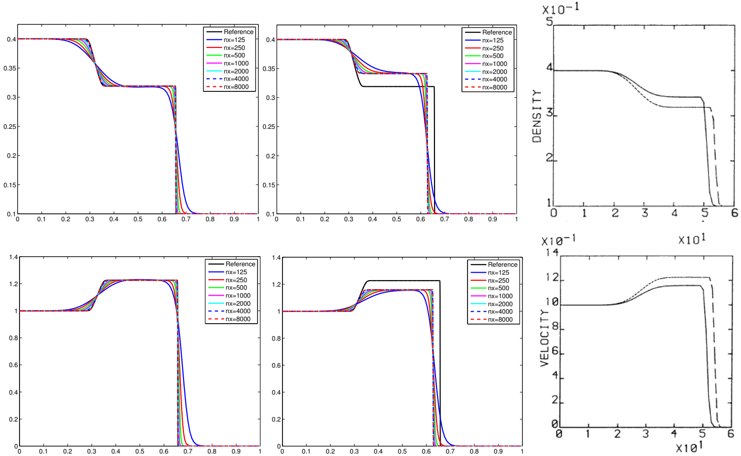

(a) Nonconservative approximation (b) Nonconservative approximation without extra term (c) From [4]

- Nonconservative systems are highly sensitive to regularization terms.
- Path conservative schemes may converge to incorrect regularization limits. (Equivalent equation has non-vanishing source term.)
- A faithful discretization of physical diffusive terms is vital for convergence.
- The recipe of entropy conservative flux + discretization of physical diffusion shows promise.

References

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