Space-time sparse discretization of linear parabolic equations

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TOC of the Talk

Introduction

Let $V \hookrightarrow H \cong H' \hookrightarrow V'$ be a Gelfand triple of separable Hilbert spaces. Consider the abstract parabolic equation in $t \in J \subset \mathbb{R}$

$$\partial_t u + Au = g, \quad u(0) = h \in H \qquad \text{(PDE)}$$

where

- $g : J \to V'$
- $A : J \to \mathcal{L}(V, V')$
- $u : J \to V$

(PDE) models

- heat conduction, e.g.

$$H_0^1(D) = V \hookrightarrow H = L^2(D) \cong H' \hookrightarrow V' = H^{-1}(D)$$

- option pricing, etc

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where

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- $A : J \to \mathcal{L}(V, V')$
- $u : J \to V$

(PDE) models

- parametric heat conduction, e.g.

$$L^2_\pi(U, H^1_0(D)) \hookrightarrow L^2_\pi(U, L^2(D)) \hookrightarrow L^2_\pi(U, H^{-1}(D))$$

- option pricing, etc

Introduction

Numerical solution of (PDE)

- many (adaptive, multi-level) Galerkin methods exist
- essentially variations on the "method of lines"

Issues

- compression of u as a function of "space-time"
- efficient algorithms for computing the *compressed* u
- provable error and complexity bounds
- design of stable finite element spaces*

* does not arise in the adaptive wavelet method [SS09]

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References I

hp-dG and -cG time-stepping

[BJ90] Ivo Babuška and Tadeusz Janik, “The h-p Version of the Finite Element Method for Parabolic Equations. II”, Numerical Methods for PDEs, 6, no. 4, 343–369, **1990**

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[HS10] Viet Ha Hoang and Christoph Schwab, “Sparse tensor Galerkin discretizations for parametric and random parabolic PDEs. I: Analytic regularity and gpc-approximation”, SAM Report 2010-11, **2010**

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Truly space-time wavelet methods

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[SS09] Christoph Schwab and Rob Stevenson, “Space-time adaptive wavelet methods for parabolic evolution problems”, Mathematics of Computation, 78, no. 267, 1293–1318, **2009**

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Introduction – continued

Class of stable FE spaces: key steps

- variational formulation of (PDE) in space-time
- test space suitably finer than ansatz space (generic)
- anisotropic (tensor product) wavelets in space-time
- sparse tensor product spaces in space-time
- derivation of a non-square (overdetermined) linear system
- corresponding normal equations are well-conditioned
- iterative solution (= residual minimization)

Variational formulation

The variational formulation, based on the bilinear form

$$b : \underbrace{L^2(J;V) \cap H^1(J;V')}_{\mathcal{U}} \times \underbrace{L^2(J;V) \times H}_{\mathcal{V}=\mathcal{V}_1\times\mathcal{V}_2} \to \mathbb{R},$$

$$b(u,v) = \int_J \langle \partial_t u + Au, v_1\rangle_{V'\times V} \mathrm{d}t + \langle u(0), v_2\rangle_H$$

and the linear functional

$$f : \underbrace{L^2(J;V) \times H}_{\mathcal{V}=\mathcal{V}_1\times\mathcal{V}_2} \to \mathbb{R}, \quad f(v) = \int_J \langle g, v_1\rangle_{V'\times V} \mathrm{d}t + \langle h, v_2\rangle_H,$$

reads: find $u \in \mathcal{U}$ s.t.

$$b(u,v) = f(v) \quad \forall v \in \mathcal{V}.$$

Variational formulation

Assume $\exists \alpha > 0, c \in \mathbb{R}$ s.t. $\forall \zeta, \eta \in V$:

- $\sup_{t \in J} ||A(t)||_{\mathcal{L}(V,V')} < \infty$
- $J \ni t \mapsto \langle A(t)\zeta, \eta \rangle_{V' \times V}$ measurable
- $\langle A(t)\eta, \eta \rangle_{V' \times V} + c||\eta||_H^2 \geq \alpha ||\eta||_V^2$ for all $t \in J$

and

- $g \in L^2(J; V')$
- $h \in H$.

Then "$b(u, \cdot) = f(\cdot)$" is well-posed [SS09]. In particular,

$$\inf_{u \in \mathcal{U} \setminus \{0\}} \sup_{v \in \mathcal{V} \setminus \{0\}} \frac{b(u,v)}{||u||_{\mathcal{U}} ||v||_{\mathcal{V}}} > 0.$$

Below assume: $c = 0$ and $A = A'$ but not $A^{-1}(t) \in K(V', V)$

Abstract stability result

Theorem

Let $\mathcal{R} \subseteq \mathcal{U}$ and $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \subseteq \mathcal{V}_1 \times \mathcal{V}_2 = \mathcal{V}$ be subspaces. Assume

- *there exists $\mathcal{K}(\mathcal{S}_1) > 0$ such that*

$$\forall z' \in \mathcal{S}_1 : \quad |z'|_{\mathcal{S}_1'} := \sup_{z \in \mathcal{S}_1 \setminus \{0\}} \frac{\langle z', z \rangle_{\mathcal{V}_1' \times \mathcal{V}_1}}{||z||_{\mathcal{V}_1}} \geq \mathcal{K}(\mathcal{S}_1) ||z'||_{\mathcal{V}_1'}$$

- $\mathcal{R} \subseteq \mathcal{S}_1$ *and* $\partial_t \mathcal{R} \subseteq \mathcal{S}_1$
- $\mathcal{R}|_{t=0} = \{u(t=0) \in H : u \in \mathcal{R}\} \subseteq \mathcal{S}_2$

Then: $\exists \gamma > 0$, only dependent on $\mathcal{K}(\mathcal{S}_1)$ and A, such that

$$\inf_{u \in \mathcal{R} \setminus \{0\}} \sup_{v \in \mathcal{S} \setminus \{0\}} \frac{b(u, v)}{||u||_{\mathcal{U}} ||v||_{\mathcal{V}}} \geq \gamma > 0.$$

Abstract stability result

Btw:

$$\forall z' \in \mathcal{S}_1 : \quad |z'|_{\mathcal{S}_1'} := \sup_{z \in \mathcal{S}_1 \setminus \{0\}} \frac{\langle z', z \rangle_{\mathcal{V}_1' \times \mathcal{V}_1}}{||z||_{\mathcal{V}_1}} \geq \mathcal{K}(\mathcal{S}_1) ||z'||_{\mathcal{V}_1'}$$

is equivalent to

$$\inf_{z' \in \mathcal{S}_1 \setminus \{0\}} \sup_{z \in \mathcal{S}_1 \setminus \{0\}} \frac{\langle z', z \rangle_{\mathcal{V}_1' \times \mathcal{V}_1}}{||z'||_{\mathcal{V}_1'} ||z||_{\mathcal{V}_1}} \geq \mathcal{K}(\mathcal{S}_1)$$

Abstract stability result

Proof, step I

- Let $u \in \mathcal{R}$.
- Set $\mathcal{Z} := \mathcal{V}_1 = L^2(J;V)$ with $||\cdot||_{\mathcal{Z}} = \langle \cdot, \cdot \rangle_{\mathcal{Z}}^{1/2}$,

$$\langle z, \widetilde{z} \rangle_{\mathcal{Z}} = \int_J \langle Az, \widetilde{z} \rangle_{V' \times V} \quad \forall z, \widetilde{z} \in \mathcal{Z}.$$

  Note: $||\cdot||_{\mathcal{Z}} \sim ||\cdot||_{L^2(J;V)} =: ||\cdot||_{\mathcal{V}_1}$ and $u \in \mathcal{R} \subseteq \mathcal{S}_1 \subseteq \mathcal{Z}$.
- Define $v_2 := u(0) \in H$ and $v_1 \in \mathcal{S}_1$ by

$$\langle v_1, \widetilde{v}_1 \rangle_{\mathcal{Z}} = \int_J \langle \partial_t u + Au, \widetilde{v}_1 \rangle_{V' \times V} \quad \forall \widetilde{v}_1 \in \mathcal{S}_1.$$

Abstract stability result

Proof, step II

- choosing $\widetilde{v}_1 = v_1$ yields fact 1

$$
\begin{aligned}
b(u,(v_1,v_2)) &= \int_J \langle \partial_t u + Au, v_1\rangle_{V'\times V} + \langle u(0), v_2\rangle_H \\
&= \langle v_1, v_1\rangle_{\mathcal{Z}} + \langle u(0), v_2\rangle_H \\
&= ||v_1||^2_{\mathcal{Z}} + ||v_2||^2_{\mathcal{V}_2}
\end{aligned}
$$

- choosing $\widetilde{v}_1 = u$ yields fact 2

$$
\begin{aligned}
\langle v_1, u\rangle_{\mathcal{Z}} &= \int_J \langle \partial_t u + Au, u\rangle_{V'\times V} \\
&= \int_J \langle Au, u\rangle_{V'\times V} + \int_J \langle \partial_t u, u\rangle_{V'\times V} \\
&= ||u||^2_{\mathcal{Z}} + \frac{1}{2}\left(||u(T)||^2_H - ||u(0)||^2_H\right)
\end{aligned}
$$

Abstract stability result

Proof, step III

Using $\partial_t u \in \partial_t \mathcal{R} \subseteq \mathcal{S}_1$ we obtain fact 3

$$
\begin{aligned}
\mathcal{K}(\mathcal{S}_1)\|\partial_t u\|_{\mathcal{V}_1'} \leq |\partial_t u|_{\mathcal{S}_1'} &= \sup_{\widetilde{v}_1 \in \mathcal{S}_1 \setminus \{0\}} \frac{\langle \partial_t u, \widetilde{v}_1 \rangle_{\mathcal{V}_1' \times \mathcal{V}_1}}{\|\widetilde{v}_1\|_{\mathcal{V}_1}} \\
&= \sup_{\widetilde{v}_1 \in \mathcal{S}_1 \setminus \{0\}} \frac{\int_J \langle \partial_t u + Au - Au, \widetilde{v}_1 \rangle_{V' \times V}}{\|\widetilde{v}_1\|_{\mathcal{V}_1}} \\
&= \sup_{\widetilde{v}_1 \in \mathcal{S}_1 \setminus \{0\}} \frac{\langle v_1, \widetilde{v}_1 \rangle_{\mathcal{Z}} - \langle u, \widetilde{v}_1 \rangle_{\mathcal{Z}}}{\|\widetilde{v}_1\|_{\mathcal{V}_1}} \\
&= \sup_{\widetilde{v}_1 \in \mathcal{S}_1 \setminus \{0\}} \frac{\langle v_1 - u, \widetilde{v}_1 \rangle_{\mathcal{Z}}}{\|\widetilde{v}_1\|_{\mathcal{Z}}} \frac{\|\widetilde{v}_1\|_{\mathcal{Z}}}{\|\widetilde{v}_1\|_{\mathcal{V}_1}} \\
&\leq \|v_1 - u\|_{\mathcal{Z}} \|\mathrm{Id}\|_{\mathcal{L}(\mathcal{V}_1, \mathcal{Z})}
\end{aligned}
$$

with $\|\mathrm{Id}\|_{\mathcal{L}(\mathcal{V}_1,\mathcal{Z})} = \operatorname{ess\,sup}_{t \in J} \|A(t)\|^{1/2}_{\mathcal{L}(V,V')}$.

Abstract stability result

Proof, wrap up

We obtain

$$
\begin{aligned}
b(u,(v_1,v_2)) &\overset{1}{=} ||v_1||_{\mathcal{Z}}^2 + ||v_2||_{\mathcal{V}_2}^2 \geq ||v_1||_{\mathcal{Z}}^2 + ||u(0)||_H^2 - ||u(T)||_H^2 \\
&\overset{2}{=} ||v_1 - u||_{\mathcal{Z}}^2 + ||u||_{\mathcal{Z}}^2 \\
&\overset{3}{\geq} \tilde{\gamma}^2 \left(||\partial_t u||_{\mathcal{V}_1'}^2 + ||u||_{\mathcal{Z}}^2 \right)
\end{aligned}
$$

with $\tilde{\gamma} = \min\{\mathcal{K}(\mathcal{S}_1)||\mathrm{Id}||_{\mathcal{L}(\mathcal{V}_1,\mathcal{Z})}^{-1}, 1\}$, and thus

$$
\begin{aligned}
b(u,(v_1,v_2)) &\geq \tilde{\gamma} \sqrt{||\partial_t u||_{L^2(J;V')}^2 + ||u||_{\mathcal{Z}}^2} \sqrt{||v_1||_{\mathcal{Z}}^2 + ||v_2||_{\mathcal{V}_2}^2} \\
&\geq \gamma \, ||u||_{L^2(J;V)\cap H^1(J;V')} \, ||(v_1,v_2)||_{L^2(J;V)\times H}
\end{aligned}
$$

with $\gamma \sim \tilde{\gamma}$.

Abstract stability result

Assumptions on $\mathcal{R} \subseteq \mathcal{U}$ and $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \subseteq \mathcal{V}_1 \times \mathcal{V}_2 = \mathcal{V}$

- there exists $\mathcal{K}(\mathcal{S}_1) > 0$ such that

$$\inf_{z' \in \mathcal{S}_1 \setminus \{0\}} \sup_{z \in \mathcal{S}_1 \setminus \{0\}} \frac{\langle z', z \rangle_{\mathcal{V}_1' \times \mathcal{V}_1}}{||z'||_{\mathcal{V}_1'} ||z||_{\mathcal{V}_1}} \geq \mathcal{K}(\mathcal{S}_1)$$

- $\mathcal{R} \subseteq \mathcal{S}_1$ and $\partial_t \mathcal{R} \subseteq \mathcal{S}_1$
- $\mathcal{R}|_{t=0} = \{u(t=0) \in H : u \in \mathcal{R}\} \subseteq \mathcal{S}_2$

can be relaxed (numerics below).

On condition $\mathcal{K}(\mathcal{S}_1) > 0$

Lemma

Assume

- $S_1 \subseteq V$ satisfies

$$\inf_{s' \in S_1 \setminus \{0\}} \sup_{s \in S_1 \setminus \{0\}} \frac{\langle s', s \rangle_{V' \times V}}{||s'||_{V'} ||s||_V} \geq \kappa(S_1) > 0$$

- similarly $\kappa(\widetilde{S}_1) > 0$ for $\widetilde{S}_1 \subseteq V$
- subspaces $E, \widetilde{E} \subseteq L^2(J)$ are orthogonal in $L^2(J)$

Then $\mathcal{S}_1 := (E \otimes S_1) + (\widetilde{E} \otimes \widetilde{S}_1) \subseteq L^2(J; V)$ satisfies

$$\mathcal{K}(\mathcal{S}_1) \geq \inf\{\kappa(S_1), \kappa(\widetilde{S}_1)\} > 0.$$

Note: generalizes to $E, \widetilde{E}, \ldots \subseteq L^2(J)$.

On condition $\mathcal{K}(\mathcal{S}_1) > 0$

Proof of Lemma: $\dim E = \dim \widetilde{E} = 1$

Let $v \in \mathcal{S}$ and $\widetilde{v} \in \widetilde{\mathcal{S}}$ (subscripts omitted). Let $E = \mathrm{span}\{e\}$ and $\widetilde{E} = \mathrm{span}\{\widetilde{e}\}$, $v = e \otimes s$ and $\widetilde{v} = \widetilde{e} \otimes \widetilde{s}$. By assumption, $\exists s' \in S$:

- $\kappa(S)\ ||s'||_{V'}\ ||s||_V \leq \langle s', s\rangle_{V' \times V}$
- w.l.o.g. $||s'||_{V'} = ||s||_V$

and similarly for some $\widetilde{s}' \in \widetilde{S}$. Therefore, $v' := e \otimes s'$ and $\widetilde{v}' := \widetilde{e} \otimes \widetilde{s}'$ are orthogonal in $\mathcal{V}$ and $\mathcal{V}'$, and moreover

$$
\begin{aligned}
||v' + \widetilde{v}'||_{\mathcal{V}'}||v + \widetilde{v}||_{\mathcal{V}} &= \sqrt{||v'||^2_{\mathcal{V}'} + ||\widetilde{v}'||^2_{\mathcal{V}'}}\sqrt{||v||^2_{\mathcal{V}} + ||\widetilde{v}||^2_{\mathcal{V}}} \\
&= \sqrt{||s'||^2_{V'} + ||\widetilde{s}'||^2_{V'}}\sqrt{||s||^2_{V} + ||\widetilde{s}||^2_{V}} \\
&= ||s'||_{V'}||s||_V + ||\widetilde{s}'||_{V'}||\widetilde{s}||_V \\
&\leq \kappa(S)^{-1}\langle v', v\rangle_{\mathcal{V}' \times \mathcal{V}} + \kappa(\widetilde{S})^{-1}\langle \widetilde{v}', \widetilde{v}\rangle_{\mathcal{V}' \times \mathcal{V}} \\
&\leq \max\{\kappa(S)^{-1}, \kappa(\widetilde{S})^{-1}\}\langle v' + \widetilde{v}', v + \widetilde{v}\rangle_{\mathcal{V}' \times \mathcal{V}}
\end{aligned}
$$

On condition $\mathcal{K}(\mathcal{S}_1) > 0$

Application of Lemma

Assume

- closed subspaces $E^{(0)} \subseteq E^{(1)} \subseteq \ldots \subseteq L^2(J)$
- closed subspaces $S_1^{(0)} \subseteq S_1^{(1)} \subseteq \ldots \subseteq V$
- $\kappa(S_1^{(\ell)}) \geq \kappa > 0$ for all $\ell \geq 0$.

Then for each $L \geq 0$ there holds

$$\mathcal{K}\left(\sum_{\ell=0}^{L} E^{(\ell)} \otimes S_1^{(L-\ell)}\right) \geq \kappa > 0.$$

Proof: using $F^{(\ell)} := \left(E^{(\ell-1)}\right)^{\perp_{L^2(J)}} \cap E^{(\ell)}$, $\ell \geq 1$ and $F^{(0)} := E^{(0)}$,

$$\sum_{\ell=0}^{L} E^{(\ell)} \otimes S_1^{(L-\ell)} = \sum_{\ell=0}^{L} \left(E^{(\ell-1)} + F^{(\ell)}\right) \otimes S_1^{(L-\ell)} = \sum_{\ell=0}^{L} F^{(\ell)} \otimes S_1^{(L-\ell)}.$$

Example

List of symbols

- Time interval $J = (0, T) \subset\subset \mathbb{R}$
- Open Lipschitz domain $D \subset\subset \mathbb{R}^d$, $d \geq 1$
- $H_0^1(D) = V \hookrightarrow H = L^2(D) \cong H' \hookrightarrow V' = H^{-1}(D)$
- Initial datum $h \in L^2(D)$
- Source term $g \in L^2(J, H^{-1}(D))$
- Conductivity $q \in L^\infty(D \times J)$ uniformly positive,

$$0 < a_{\min} \leq \operatorname*{ess\,inf}_{D \times J} q \leq \operatorname*{ess\,sup}_{D \times J} q \leq a_{\max} < \infty$$

Example

(PDE) in strong form

Find $u : D \times J \to \mathbb{R}$ such that

$$\partial_t u(x,t) - \nabla \cdot (q(x,t)\nabla u(x,t)) = g(x,t), \quad (x,t) \in D \times J$$

with initial condition

$$u(x,0) = h(x), \quad x \in D$$

and boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial D \times J$$

Example

(PDE) in weak form

Find $u \in \mathcal{U} = L^2(J, H_0^1(D)) \cap H^1(J, H^{-1}(D))$ such that

$$\int_J \int_D (v_1 \partial_t u + q \nabla u \cdot \nabla v_1)\, \mathrm{d}x\, \mathrm{d}t = \int_J \int_D g v_1 \mathrm{d}x\, \mathrm{d}t$$

and

$$\int_D u(0) v_2 \mathrm{d}x = \int_D h v_2 \mathrm{d}x$$

for all $v = (v_1, v_2) \in \mathcal{V} = L^2(J, H_0^1(D)) \times L^2(D)$,
where q, u, v_1, g depend on (x, t), and $u(0)$, h, v_2 on x

Example

(PDE) in weak form

With bilin. form $b : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$ and lin. functional $f : \mathcal{V} \to \mathbb{R}$,

$$b(u, v) = \int_J \int_D (v_1 \partial_t u + q \nabla u \cdot \nabla v_1) \,\mathrm{d}x \mathrm{d}t + \int_D u(0) v_2 \mathrm{d}x$$

and

$$f(v) = \int_J \int_D g v_1 \mathrm{d}x \mathrm{d}t + \int_D h v_2 \mathrm{d}x,$$

the variational formulation reads: find

$$u \in \mathcal{U} : \quad b(u, v) = f(v) \quad \forall v \in \mathcal{V}$$

Example

Choice of temporal basis

Let $\Theta_L = \{\theta_\lambda : \lambda \in \mathcal{I}_\Theta : |\lambda| \leq L\} \subset L^2(J)$ be

- a Riesz basis for its span uniformly in level $L \geq 0$:

$$||\Theta_L^\top \mathbf{c}||_{L^2(J)} \sim ||\mathbf{c}||_2 \quad \forall \mathbf{c} \in \mathbb{R}^{\#\Theta_L}$$

- defined w.r.t. an equidistant partition of $J = (0, T)$ with width $h \sim 2^{-L}$
- piecewise linear continuous
- s.t. $\bigcup_{\ell \geq 0} \Theta_\ell$ rescales to a Riesz basis for $H^1(J)$

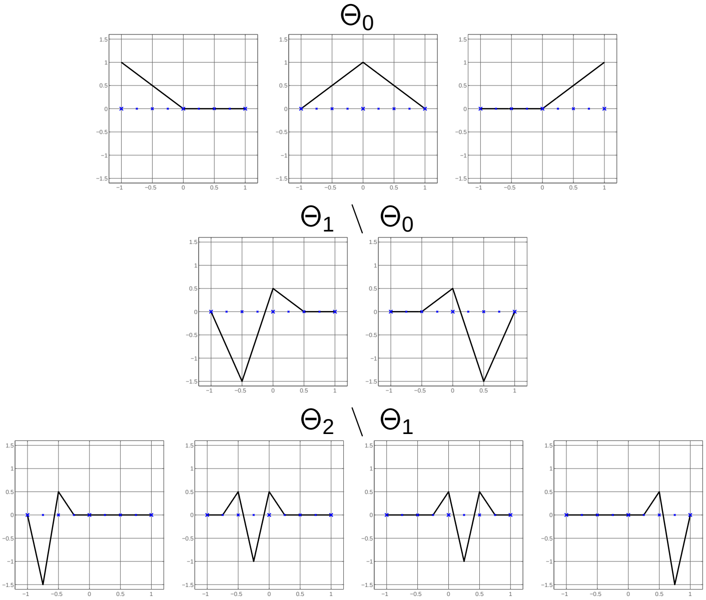

Figure: Piecewise linear continuous biorthogonal B-spline wavelets

Example

Choice of spatial basis

Let $\Sigma_L = \{\sigma_\mu : \mu \in \mathcal{I}_\Sigma : |\mu| \leq L\} \subset L^2(D)$ be

- a Riesz basis for its span uniformly in level $L \geq 0$
- defined w.r.t. a sequence of quasi-uniform nested triangulations of $D \subset \mathbb{R}^d$ with mesh width $h \sim 2^{-L}$
- s.t. $\bigcup_{\ell \geq 0} \Sigma_\ell$ rescales to a Riesz basis for $H_0^1(D)$
- (s.t. $\bigcup_{\ell \geq 0} \Sigma_\ell$ rescales to a Riesz basis for $H^{-1}(D)$)

Example

Choice of ansatz/test spaces I

Define full tensor product

- ansatz space $R_L \subset \mathcal{U}$ spanned by

$$\{\theta_\lambda \otimes \sigma_\mu : |\lambda| \leq L, |\mu| \leq \rho L\}$$

- test space $S_L \subset \mathcal{V}$ spanned by

$$\{\theta_\lambda \otimes \sigma_\mu : |\lambda| \leq L+1, |\mu| \leq \rho L\} \times \{\sigma_\mu : |\mu| \leq \rho L\}$$

where $\rho > 0$ is the degree of anisotropy in the refinement

Example

Choice of ansatz/test spaces II

Define sparse tensor product

- ansatz space $R_L \subset \mathcal{U}$ spanned by

$$\{\theta_\lambda \otimes \sigma_\mu : |\lambda| + \rho^{-1}|\mu| \leq L\}$$

- test space $S_L \subset \mathcal{V}$ spanned by

$$\{\theta_\lambda \otimes \sigma_\mu : \min\{|\lambda| - 1, 0\} + \rho^{-1}|\mu| \leq L\} \times \{\sigma_\mu : |\mu| \leq \rho L\}$$

where $\rho > 0$ is the degree of anisotropy in the refinement

Example

Formulation via residual minimization (FTP case)

The discrete version of the variational formulation: find

$$\widetilde{u}_L := \arg\min_{w \in R_L} \max_{v \in S_L \setminus \{0\}} \frac{|b(w,v) - f(v)|}{||v||_{\mathcal{V}}}$$

motivates, via Riesz bases $\bigcup_{\ell \geq 0} \Theta_\ell$ and $\bigcup_{\ell \geq 0} \Sigma_\ell$, the definition

$$\mathbf{u}_L := \arg\min_{\mathbf{w} \in \mathbb{R}^{\dim R_L}} ||\mathbf{B}_L \mathbf{w} - \mathbf{f}_L||_2 \qquad \text{(argmin)}$$

which defines $u_L = [\Theta_L \otimes \Sigma_L]^\top \mathbf{u}_L \in R_L$. In general $u_L \neq \widetilde{u}_L$.

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Example

Formulation via residual minimization (FTP case)

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which defines $u_L = [\Theta_L \otimes \Sigma_L]^\top \mathbf{u}_L \in R_L$. In general $u_L \neq \tilde{u}_L$.

Example

Quasi-optimality

Theorem

The least-squares Galerkin solution u_L exists, is unique and converges quasi-optimally uniformly in $L \geq 0$: there exists $C > 0$ such that $\forall L \geq 0$

$$||u_L - u||_{\mathcal{U}} \leq C \inf_{w_L \in R_L} ||w_L - u||_{\mathcal{U}}.$$

Example

Numerics I – smooth solution, FTP

Experimental set-up

- $D = (-1, 1) \subset \mathbb{R}$, $J = (0, T)$ with $T = 2$
- conductivity in (PDE) is $q \equiv 1$
- exact solution

$$u(x,t) = e^{-t} \cos \frac{x\pi}{2}, \quad (x,t) \in D \times J,$$

- piecewise linear continuous B-spline wavelet basis
- degree of anisotropy in refinement $\rho = 1$

Example

Numerics I – smooth solution, FTP

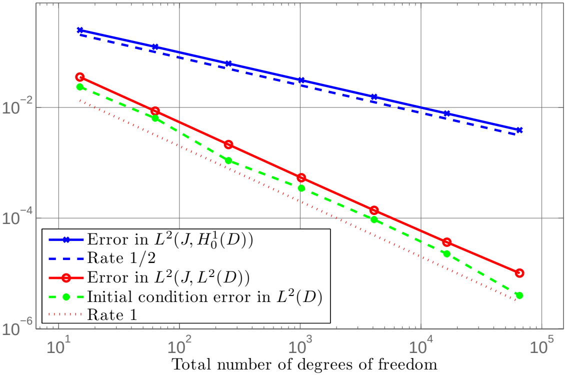

Figure: Convergence of the Galerkin solution u_L to a smooth u

Example

Numerics I – smooth solution, FTP

The system (argmin) is

- sparse
- well-conditioned

which suggests the use of an iterative least squares algorithm[1]

Level L	1	2	$\cdots$	5	6	7
ndof + 1	2^4	2^6	$\cdots$	2^{12}	2^{14}	2^{16}
# iterations	11	41	$\cdots$	315	442	574
$\log_{10}$ rel. res.	-1.51	-2.08	$\cdots$	-3.54	-4.02	-4.5

Table: The number of MATLAB's `lsqr` iterations with tolerance 10^{-10}, the corresponding number of degrees of freedom, and the relative residual on various discretization levels L for u_L

[1] Paige and Saunders, SIAM J. Numer. Anal, 12, **1975**

Example

Numerics II – rough solution, FTP

Experimental set-up

- $D = (-1, 1) \subset \mathbb{R}$, $J = (0, T)$ with $T = 2$
- source $g \equiv 1$, initial condition $h \equiv 0$
- conductivity in (PDE) is $q(x, t) = 1 - \frac{1}{2}\,\mathrm{sign}\,(2 + x - 2t)$
- piecewise linear continuous B-spline wavelet basis
- degree of anisotropy in refinement $\rho = 1$
- reference solution on level $L = 7$

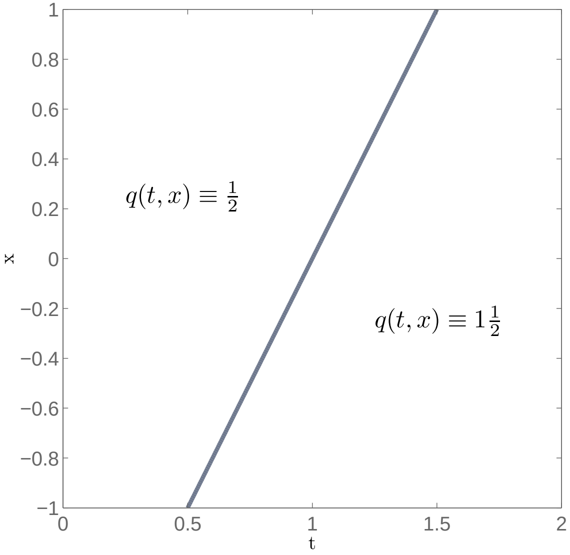

Figure: Conductivity q

Example

Numerics II – rough solution, FTP

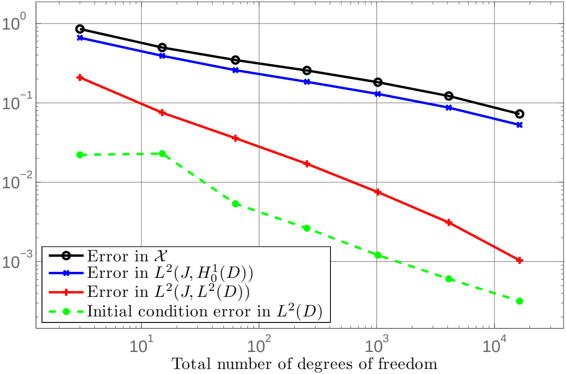

Figure: Convergence[2] of the Galerkin solution u_L to u_7

[2] $\mathcal{X} \equiv \mathcal{U}$

Example

Numerics III – FTP vs STP

Experimental set-up

- $D = (-1, 1) \subset \mathbb{R}$, $J = (0, T)$ with $T = 2$
- conductivity $q \equiv 1$, initial condition $h \equiv 0$
- source $g(x, t) = \cos(x + \cos \frac{\pi t}{2})\ \sin^2 \frac{\pi t}{2}$
- piecewise linear continuous B-spline wavelet basis
- degree of anisotropy in refinement $\rho = 1$
- reference FTP solution on level $L = 7$

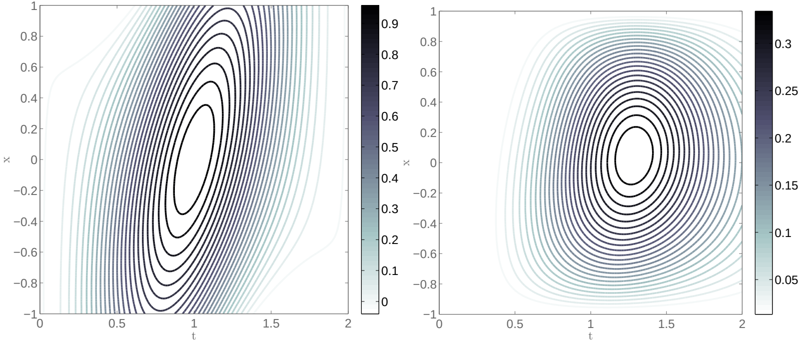

Figure: Source g (**left**) and solution u (**right**)

Example

Numerics III – FTP vs STP

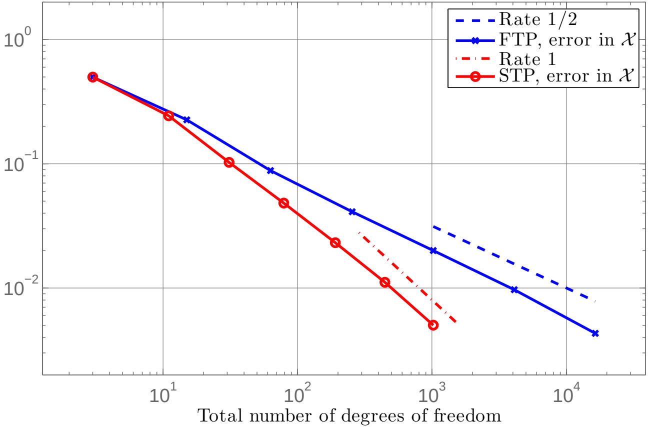

Figure: Convergence[3] of the Galerkin solution: STP vs FTP

[3] $\mathcal{X} \equiv \mathcal{U}$

Outro

Review

- sparse tensor formulation of (PDE) in space-time
- inf-sup condition for the associated bilinear form
- quasi-optimality of the Galerkin least squares solution
- examples

In progress

- sparse discretization of parametric parabolic equations
- weighted spaces for corners of the space-time cylinder

Thank You!