The AL Basis for the Solution of Elliptic Problems in Heterogeneous Media

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Pro*Doc Retreat Disentis, 13-15th August 2014

Problem Formulation

$\Omega \subset \mathbb{R}^2$ a bounded Lipschitz domain with piecewise analytic boundary

For a given function $f \in L^2(\Omega)$, we are seeking $u \in H_0^1(\Omega)$ such that

$$a(u, v) := \int_\Omega \langle A \nabla u, \nabla v \rangle = \int_\Omega f v =: F(v) \qquad \forall v \in H_0^1(\Omega).$$

The diffusion matrix $A \in L^\infty(\Omega, \mathbb{R}^{2\times 2}_{sym})$ is uniformly elliptic, i.e.

$$0 < \alpha(A, \Omega) := \operatorname*{ess\,inf}_{x \in \Omega} \inf_{v \in \mathbb{R}^2 \setminus \{0\}} \frac{\langle A(x) v, v \rangle}{\langle v, v \rangle}$$

$$\infty > \beta(A, \Omega) := \operatorname*{ess\,sup}_{x \in \Omega} \sup_{v \in \mathbb{R}^2 \setminus \{0\}} \frac{\langle A(x) v, v \rangle}{\langle v, v \rangle}.$$

Partition of Unity Finite Element Method (PUM)

1. Let $\omega_1, \ldots, \omega_N$ be open sets satisfying
   - $\Omega = \bigcup_{i=1}^{N} \omega_i$
   - $\exists M \in \mathbb{N}$ s.t. $\forall x \in \Omega$ we have $\#\{i \mid x \in \omega_i\} \leq M$
2. Construct local approximation spaces V_i, i.e. on each patch ω_i let $V_i \subset H_0^1(\Omega)|_{\omega_i}$ be a space of functions by which the solution $u|_{\omega_i}$ can be approximated well.
3. Choose partition of unity functions $(\varphi_i)_{i=1}^{N}$ such that
   - $\operatorname{supp} \varphi_i \subset \overline{\omega_i}$
   - $\sum_i \varphi_i = 1$ on Ω
   - $\|\varphi_i\|_{L^\infty(\Omega)} \leq C$
   - $\|\nabla \varphi_i\|_{L^\infty(\Omega)} \leq \frac{C}{\operatorname{diam} \omega_i}$
4. Define the global finite element space V by

$$V := \sum_i \varphi_i V_i.$$

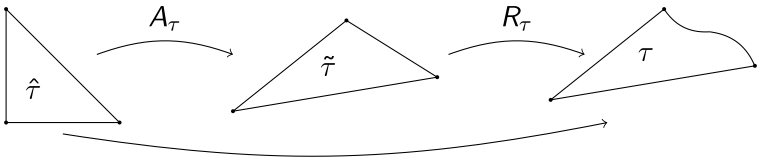
Properties of PUM and Potential Applications

- A priori knowledge of the problem under consideration can be incorporated in the construction of the local approximation spaces.
- The global space V inherits the approximation properties of the local spaces V_i, i.e. the function u can be approximated on Ω by functions of V as well as the functions $u|_{\omega_i}$ can be approximated in the local spaces V_i.
- V inherits the smoothness of the partition of unity $(\varphi_i)_{i=1}^N$. In particular, the smoothness of the partition of unity enforces the conformity of the global space V.

Potential Applications:

problems where the solution is rough or highly oscillatory, e.g. elasticity equations for laminated materials, heterogeneous materials, or the Helmholtz equation for large wave number

The Triangulation $\mathcal{G}$



Assumption (quasi-uniform regular triangulation)

Each element map F_τ can be written as $F_\tau = R_\tau \circ A_\tau$, where A_τ is an affine map (containing the scaling by h_τ) and R_τ is an h_τ-independent analytic map. Let $\tilde{\tau} := A_\tau(\hat{\tau})$. The maps R_τ and A_τ satisfy for shape regularity constants C_{affine}, C_{metric}, $\gamma > 0$ independent of H:

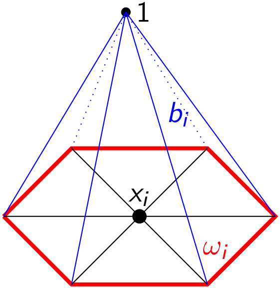
$$\|A'_\tau\|_{L^\infty(\hat{\tau})} \leq C_{affine} H, \qquad \|(A'_\tau)^{-1}\|_{L^\infty(\hat{\tau})} \leq C_{affine} H^{-1}$$

$$\|(R'_\tau)^{-1}\|_{L^\infty(\tilde{\tau})} \leq C_{metric}, \qquad \|\nabla^n R_\tau\|_{L^\infty(\tilde{\tau})} \leq C_{metric} \gamma^n n! \quad \forall n \in \mathbb{N}_0.$$

Approximation Space and Basis Functions

Define

$$S := \{u \in H_0^1(\Omega) \mid \forall \tau \in \mathcal{G} : u|_\tau \circ F_\tau \in \mathbb{P}_1\}.$$



$(b_i)_{i=1}^n$ usual $\mathbb{P}_1$ basis

$\omega_i :=$ supp b_i

conforming Galerkin method: Find $u_S \in S$ such that

$$a(u_S, v) = F(v) \quad \forall v \in S.$$

If A, f and Ω are sufficiently smooth such that the problem is H^2-regular, then

$$\|u - u_S\|_{H^1(\Omega)} \leq CH\|f\|_{L^2(\Omega)}.$$

Goal:

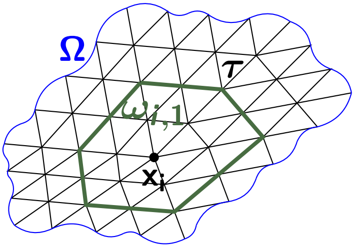
Construct $V_{AL} \subset H_0^1(\Omega)$ such that the linear convergence rate is preserved for heterogeneous and/or highly oscillatory coefficients.

Let $L_\Omega^{-1} : L^2(\Omega) \to H_0^1(\Omega)$ denote the solution operator: Given $f \in L^2(\Omega)$, find $u \in H_0^1(\Omega)$ such that

$$a(u,v) := \int_\Omega \langle A\nabla u, \nabla v \rangle = \int_\Omega fv =: F(v).$$

1. $V_{AL} := \operatorname{span}\{L_\Omega^{-1}(\chi_\tau) \mid \tau \in \mathcal{G}\}$
   - global basis functions
   - very large overlap
   - not computable

2. $V_i := \operatorname{span}\{b_i L_\Omega^{-1}(\chi_\tau) \mid \tau \in \mathcal{G}\}$
   $V_{AL} := V_1 + V_2 + \cdots + V_n$
   + local basis functions
   - very large overlap
   - not computable



$V_i^{near} := \text{span}\{b_i L_\Omega^{-1}(\chi_\tau) \mid \tau \subset \omega_{i,1}\}$
$X_i^{far} := \text{span}\{L_\Omega^{-1}(\chi_\tau)|_{\omega_{i,1}} \mid \tau \subset \Omega \backslash \omega_{i,1}\}$
$\tilde{V}_i^{far}$ low dim. approximation of X_i^{far}
$V_i^{far} := \{b_i v \mid v \in \tilde{V}_i^{far}\}$

3. $V_i := V_i^{near} + V_i^{far}$
   $V_{AL} := V_1 + V_2 + \cdots + V_n$
   - \+ local basis functions
   - \+ small overlap
   - – not computable

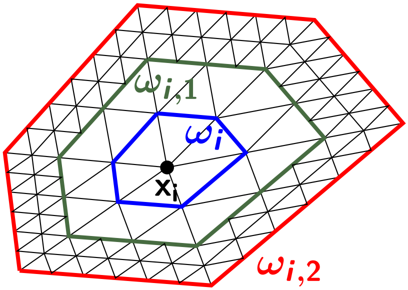
4. $L_\Omega^{-1} \leftarrow L_{\Omega,h}^{-1}$
   - \+ local basis functions
   - \+ small overlap
   - \+ computable
   - – not efficient

The Mesh and Local Solution Operator

$\omega_{i,0} := \omega_i = \text{supp}\, b_i$
$\omega_{i,j+1} := \bigcup\{\bar{\tau} \mid \tau \in \mathcal{G} \text{ and } \omega_{i,j} \cap \bar{\tau} \neq \emptyset\}$
$\omega_i^{far} := \omega_{i,2} \backslash \overline{\omega_{i,1}}$

$\mathcal{G}_{i,j} := \{\tau \in \mathcal{G} \;:\; \tau \subset \omega_{i,j}\}$
$\mathcal{G}_i^{far} := \mathcal{G}_{i,2} \backslash \mathcal{G}_{i,1}$

$\mathcal{R}^1(\mathcal{G}_i^{far})$ refinement of $\mathcal{G}_i^{far}$
$\mathcal{R}^t(\mathcal{G}_i^{far}) := \mathcal{R}^1(\mathcal{R}^{t-1}(\mathcal{G}_i^{far}))$



For a subdomain $\omega_{i,2} \subset \Omega$ let $L^{-1}_{\omega_{i,2}} : L^2(\omega_{i,2}) \to H^1_0(\omega_{i,2})$ denote the local solution operator: Given $g \in L^2(\omega_{i,2})$, find $u \in H^1_0(\omega_{i,2})$ such that

$$a(u, v) := \int_{\omega_{i,2}} \langle A\nabla u, \nabla v\rangle = \int_{\omega_{i,2}} gv =: G(v).$$

Construction of Local PUM Spaces I

$S_H := \text{span}\{\chi_\tau \mid \tau \in \mathcal{G}\}, \qquad S_H \subset S_h \subset H_0^1(\Omega)$
$S_{i,2,h} := \{u|_{\omega_{i,2}} \mid u \in S_h \wedge \text{supp}\, u \subset \omega_{i,2}\}$

Nearfield ($\tau \in \mathcal{G}_{i,1}$):
Find $\tilde{B}_{i,\tau} \in S_{i,2,h}$ such that

$$\int_{\omega_{i,2}} \langle A \nabla \tilde{B}_{i,\tau}, \nabla v \rangle = \int_{\omega_{i,2}} \chi_\tau v \quad \forall v \in S_{i,2,h}.$$

$B_{i,\tau} := b_i \tilde{B}_{i,\tau} \qquad V_i^{near} := \text{span}\{B_{i,\tau} \mid \tau \in \mathcal{G}_{i,1}\}$

Farfield ($\tau \in \mathcal{R}^t(\mathcal{G}_i^{far})$, $t \sim \log \frac{1}{H}$):
Find $\tilde{B}_{i,\tau} \in S_{i,2,h}$ such that

$$\int_{\omega_{i,2}} \langle A \nabla \tilde{B}_{i,\tau}, \nabla v \rangle = \int_{\omega_{i,2}} \chi_\tau v \quad \forall v \in S_{i,2,h}.$$

Construction of Local PUM Spaces II

$X_i^{far} := \operatorname{span}\{\tilde{B}_{i,\tau}|_{\omega_{i,1}} \ : \ \tau \in \mathcal{R}^t(\mathcal{G}_i^{far})\}$

The functions in X_i^{far} are locally L-harmonic on $\omega_{i,1}$, i.e. any $v \in X_i^{far}$ satisfies

$$\int_{\omega_{i,1}} \langle A\nabla v, \nabla w\rangle = 0 \quad \forall w \in S_{i,1,h} := \{w \in S_h \ : \ \operatorname{supp} w \subset \omega_{i,1}\}.$$

X_i^{far} can be approximated by a low dimensional space. Let $\tilde{V}_i^{far}$ be the low dimensional approximation of X_i^{far}. Then we set

$$V_i^{far} := \{b_i v \,|\, v \in \tilde{V}_i^{far}\}.$$

Finally we define

$$V_{AL} := (V_1^{near} + V_1^{far}) + \cdots + (V_n^{near} + V_n^{far}).$$

Approximation of X_i^{far}

Lemma (Bebendorf, Hackbusch 2003)

Let $D \subset \Omega$ and $X(D)$ the space of locally L-harmonic functions on D. Furthermore, let $D_2 \subset D$ be a convex domain such that

$$\operatorname{dist}(D_2, \partial D) \geq \operatorname{diam}(D_2) > 0.$$

Then for any $M > 1$ there is a subspace $W \subset X(D_2)$ so that

$$\inf_{w \in W} \|u - w\|_{L^2(D_2)} \leq \frac{1}{M} \|u\|_{L^2(D)} \quad \forall u \in X(D)$$

and

$$\dim W \leq c^d \lceil \log M \rceil^{d+1} \tag{1}$$

where d is the spatial dimension and c only depends on α and β.

Error Analysis I

1. Let $S := \operatorname{span}\{\chi_\tau \mid \tau \in \mathcal{G}\}$. The approximation of u is given by

$$u_{AL} = L_\Omega^{-1} P_S f$$

where $P_S : L^2(\Omega) \to S$ is the L^2-orthogonal projection onto S.

2.

$$u_{AL} = \sum_{i=1}^{n} b_i L_\Omega^{-1} P_S f = L_\Omega^{-1} P_S f$$

3.

$$f_i^{near} := \sum_{x_j \in \mathring{\omega}_{i,1}} (P_S f)(x_j) b_j \quad \text{and} \quad f_i^{far} := \sum_{x_j \in \mathring{\Omega} \setminus \mathring{\omega}_{i,1}} (P_S f)(x_j) b_j$$

$$L_\Omega^{-1} P_S f = \sum_{i=1}^{n} \underbrace{b_i L_\Omega^{-1} f_i^{near}}_{u_i^{near}} + \sum_{i=1}^{n} b_i \underbrace{L_\Omega^{-1} f_i^{far}}_{u_i^{far}}.$$

Error Analysis II

Lemma (Caccioppoli inequality)

Let $u \in X(D)$ *and let* $K \subseteq D$ *be a domain with* $\operatorname{dist}(K, \partial D) > 0$*. Then we have* $u|_K \in H^1(K)$ *and*

$$\|\nabla u\|_{L^2(K)} \leq \sqrt{\frac{\beta}{\alpha}} \frac{4}{\operatorname{dist}(K, \partial D)} \|u\|_{L^2(D)}.$$

There exists $\tilde{u}_i^{far} \in \tilde{V}_i^{far}$ such that

$$\|u_i^{far} - \tilde{u}_i^{far}\|_{H^m(\omega_i)} \leq CH^{3-m} \|\nabla L_\Omega^{-1} f_i^{far}\|_{L^2(\omega_{i,1})} \quad m = 0, 1.$$

$$u_{AL} := \sum_{i=1}^{n} u_i^{near} + \sum_{i=1}^{n} b_i \tilde{u}_i^{far} \in V_{AL}$$

Error Analysis III

$$\begin{aligned}\|u - u_{AL}\|_{H^1(\Omega)} &\leq \|u - L_\Omega^{-1} P_S f\|_{H^1(\Omega)} + \|L_\Omega^{-1} P_S f - u_{AL}\|_{H^1(\Omega)} \\ &\leq CH\|f\|_{L^2(\Omega)} + \left\| \sum_{i=1}^{n} b_i (u_i^{far} - \tilde{u}_i^{far}) \right\|_{H^1(\Omega)}\end{aligned}$$

4. Replace L_Ω^{-1} by $L_{\Omega,h}^{-1}$.

Assumption

Let $f \in H^{p-1}(\Omega)$ for some $p \in \mathbb{N}$.

$$\sup_{f \in H^{p-1}(\Omega) \setminus \{0\}} \inf_{v \in S_h} \frac{\|L_\Omega^{-1} f - v\|_{H^1(\Omega)}}{\|f\|_{H^{p-1}(\Omega)}} \leq CH$$

Céa's lemma implies

$$\|L_\Omega^{-1} f - L_{\Omega,h}^{-1} f\|_{H^1(\Omega)} \leq CH\|f\|_{L^2(\Omega)}.$$

Error Analysis IV

Corollary (Peterseim, Sauter 2012)

Let $A \in C^p(\Omega, \mathbb{R}^{d\times d}_{sym})$ *for some* $p \in \mathbb{N}$ *be uniformly elliptic and assume* $f \in H^{p-1}(\Omega)$*. Further let the boundary* $\partial\Omega$ *be of class* C^p*. Assume that the coefficient* A *satisfies*

$$\frac{1}{q!}\|\nabla^q A\|_{L^\infty(\Omega)} \leq C\epsilon^{-q}$$

for some (small) $\epsilon > 0$ *and for all* $1 \leq q \leq p$*. Let* p *and* h *be chosen such that*

$$p = \left\lceil \frac{\log h}{\log(C_9' h/\epsilon)} \right\rceil \quad \textit{and} \quad C_9' h < \epsilon$$

holds. Then the Galerkin discretization with the corresponding hp-finite element space S_h^p *has a unique solution* u_h *which converges linearly*

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch\|f\|_{H^{p-1}(\Omega)}.$$

Error Analysis V

5. We set $u_i := \chi_i(u - \bar{u}_i)$ and observe that

$$u_i = L^{-1}_{\omega_{i,2}}(g_i)$$

with

$$g_i = \begin{cases} f & \text{in } \omega_{i,1} \\ \chi_i f - 2\langle A\nabla\chi_i, \nabla u\rangle - (u - \bar{u}_i)\,\mathrm{div}\,(A\nabla\chi_i) & \text{in } \omega_i^{far}. \end{cases}$$

We set

$$g_i^{near} := \begin{cases} f & \text{in } \omega_{i,1} \\ 0 & \text{in } \omega_i^{far} \end{cases}$$

$$g_i^{far} := \begin{cases} 0 & \text{in } \omega_{i,1} \\ \chi_i f - 2\langle A\nabla\chi_i, \nabla u\rangle - (u - \bar{u}_i)\,\mathrm{div}\,(A\nabla\chi_i) & \text{in } \omega_i^{far}. \end{cases}$$

Cutoff Function

Let $Q \in (2, \infty)$ be fixed. For $\frac{Q'}{3} \leq q \leq \frac{Q}{3}$

- $\|\chi_i\|_{L^q(\omega_i^{far})} \leq C H_i^{\frac{2}{q}}$
- $\|\nabla \chi_i\|_{L^q(\omega_i^{far})} \leq \frac{C}{H_i} H_i^{\frac{2}{q}} = C H_i^{\frac{2-q}{q}}$
- $\|\operatorname{div}(A \nabla \chi_i)\|_{L^q(\omega_i^{far})} \leq \frac{C}{H_i^2} H_i^{\frac{2}{q}} = C H_i^{\frac{2-2q}{q}}$
- $\chi_i|_{\overline{\omega_{i,1}}} = 1$
- $\chi_i|_{\partial \omega_{i,2}} = 0.$

Ansatz mollifier: $\varphi(x) := x^2(3 - 2x)$
Let $\hat{\eta} \in H^1(\hat{\omega}_i^{far})$ be such that

$$\operatorname{div}(\hat{A} \nabla \hat{\eta}) = 0 \text{ in } \hat{\omega}_i^{far}$$

$$\hat{\eta} = \begin{cases} 0 \text{ on } \partial \hat{\omega}_{i,2} \\ 1 \text{ on } \partial \hat{\omega}_{i,1} \end{cases}$$

Finally we set $\hat{\chi} := \varphi(\hat{\eta})$.

$W^{1,p}$-Regularity of Poisson Problem with L^∞-Coefficient

Given $F \in W^{-1,q}(\Omega)$ for some $q \in [2, \infty)$, find $w \in H_0^1(\Omega)$ such that

$$\int_\Omega \langle \nabla w, \nabla v \rangle = F(v) \quad \forall v \in H_0^1(\Omega).$$

Let $Q > 2$ be fixed. For a bounded domain Ω with $\partial\Omega \in C^1$ there exists a constant $K_Q > 1$ depending only on Ω and Q such that

$$\|\nabla w\|_{L^Q(\Omega)} \leq K_Q \|F\|_{W^{-1,Q}(\Omega)}.$$

Find $u \in H^1(\Omega)$ such that

$$\begin{aligned} \operatorname{div}(A \nabla u) &= 0 \quad \text{on } \Omega \\ u &= h \quad \text{on } \partial\Omega. \end{aligned} \tag{2}$$

Theorem

Let $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded domain and let $\partial\Omega \in C^1$. Further let $Q \in (2, \infty)$ be fixed and $Q' = \frac{Q}{Q-1}$. Assume that $A \in L^\infty\left(\Omega, \mathbb{R}^{d\times d}_{sym}\right)$ is uniformly elliptic and $\frac{\alpha}{\beta} \in \left[1 - \frac{1}{K_Q}, 1\right)$. Then equation (2) is uniquely solvable in $W^{1,p}(\Omega)$ for each $h \in W^{1-1/p,p}(\partial\Omega)$ with $Q' \leq p \leq Q$ and the solution satisfies

$$\|\nabla u\|_{L^p(\Omega)} \leq C \|h\|_{W^{1-1/p,p}(\partial\Omega)}$$

with a constant C depending on β, p, and the domain Ω.

Main Result

Assumption

$$\sup_{f \in L^2(\omega_{i,2})\setminus\{0\}} \inf_{v \in S_{i,2,h}} \frac{\|L^{-1}_{\omega_{i,2}} f - v\|_{H^1(\omega_{i,2})}}{\|f\|_{L^2(\omega_{i,2})}} \leq C h_i,$$

where h_i denotes the mesh width of the refined mesh $\mathcal{R}^t(\mathcal{G}_i^{far})$ ($t \sim \log \frac{1}{H}$).

The control parameters for the AL-basis can be chosen in such a way that for every $f \in L^\infty(\Omega)$ and $A \in L^\infty(\Omega, \mathbb{R}^{2\times 2}_{sym})$ such that $\frac{\alpha}{\beta} \in [1 - C, 1)$ with $C = O(1)$ the estimate

$$\|u - u_{AL}\|_{H^1(\Omega)} \leq CH\|f\|_{L^\infty(\Omega)}$$

holds and for the dimension we have

$$\dim V_{AL} \leq CH^{-2} \log^3 \frac{1}{H}.$$

Thank you for your attention!