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Seminar for Applied Mathematics SAM

The interface problem for the Wave Equation: Asymptotics and FEM discretization

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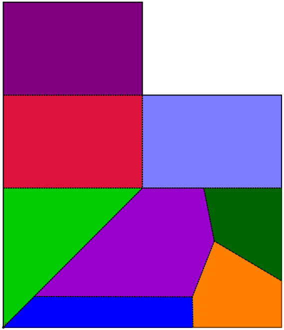
Interface problems

- Polygonal domain $\Omega \subseteq \mathbb{R}^2$.
- Decomposition into polygonal domains $\Omega_j \subseteq \Omega$.
  Boundaries: $\partial\Omega_j = \bigcup_q \bar{\gamma}_{jq}$.
- Piecewise constant wave speed $c(x)$, $c|_{\Omega_j} \equiv c_j \in \mathbb{R}_{>0}$.
- Set of edges $\mathcal{E} = \mathcal{D} \cup \mathcal{N} \cup \mathcal{I}$.
  Dirichlet boundary: $\mathcal{D} \subseteq \mathcal{E}$.
  Neumann boundary: $\mathcal{N} \subseteq \mathcal{E}$.
  Interior edges:
  $\mathcal{I} \ni \gamma_{jq} \iff \gamma_{jq} = \gamma_{j'q'} \in \mathcal{I}$.
- Time horizon $0 < T < \infty$, interval $I := (0, T)$.



Wave Equation: Transmission problem

For a function $w(x)$, denote $w_j := w|_{\Omega_j}$ and $w_{jq} := w|_{\gamma_{jq}}$.

PDE in each Ω_j:

$$\partial_t^2 u_j(x,t) - \nabla \cdot (c_j(x) \nabla u_j(x,t)) = f_j(x,t), \quad (x,t) \in \Omega_j \times I \ .$$

Transmission conditions at $\mathcal{I} \ni \gamma_{jq} = \gamma_{j'q'}$:

$$\begin{aligned} u_{jq}(x,t) &= u_{j'q'}(x,t), \quad x \in \gamma_{jq}, t \in I \ , \\ c_j \partial_{\boldsymbol{\nu}_{jq}} u_{jq}(x,t) &= -c_{j'} \partial_{\boldsymbol{\nu}_{j'q'}} u_{j'q'}(x,t), \quad x \in \gamma_{jq}, t \in I \ . \end{aligned}$$

Boundary conditions:

$$\begin{aligned} u_{jq}(x,t) &= 0, \quad \gamma_{jq} \in \mathcal{D}, t \in I \ , \\ \partial_{\boldsymbol{\nu}_{jq}} u(\mathbf{x},t) &= 0, \quad \gamma_{jq} \in \mathcal{N}, t \in I \ . \end{aligned}$$

Initial conditions:

$$\begin{aligned} u(\cdot,0) &\equiv u^0, \\ \partial_t u(\cdot,0) &\equiv u^1 \ . \end{aligned}$$

Weak Form

Weak form:
Let $V := \{u \in H^1(\Omega) \,|\, u|_{\Gamma_D \times \bar{I}} \equiv 0\}$.

Find $u \in H^1(I;V) \cap H^2(I;V^*)$, s.t. $\forall t \in I, v \in V$:

$$(\partial_t^2 u(\cdot,t), v)_{L^2} + \underbrace{\sum_j \int_{\Omega_j} c_j \nabla u(\cdot,t) \cdot \nabla v \, dx}_{=:a(u(\cdot,t),v)} = (f(\cdot,t), v)_{L^2} \ ,$$

$$(u(\cdot,0) - u^0, v)_{L^2} = 0 \ ,$$

$$(\partial_t u(\cdot,0) - u^1, v)_{L^2} = 0 \ .$$

For $c_j > c_0 > 0$: Operator $-\nabla \cdot (c \nabla u)$ + t.c. and b.c.:

regular elliptic transmission problem.

Nicaise S. and Sändig, A.-M.: *General interface problems I.* Math. methods appl. sci., 1994

Well-posedness of the weak form

Theorem

If $u^0 \in L^2(\Omega)$, $u^1 \in V$, $f \in C^0(\bar{I}; V)$, *there is a unique solution*

$$u \in C^2(\bar{I}; V^*) \cap C^1(\bar{I}; L^2(\Omega)) \cap C^0(\bar{I}; V) \ .$$

If $f \equiv 0$, *the following energy is preserved:*

$$E(t) := \frac{1}{2}\|\partial_t u(\cdot, t)\|_{L^2(\Omega)} + \frac{1}{2} a(u(\cdot, t), u(\cdot, t)) \ .$$

Proof sketch:

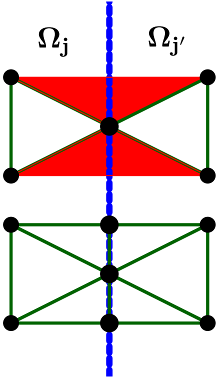
- $a(\cdot, \cdot)$ is V-elliptic and symmetric on V.
- Spectral theorem diagonalizes weak form to ODEs $\ddot{\alpha}_k(t) + \lambda_k \alpha_k(t) = f_k(t)$.
- Check convergence of series in respective norms.

FEM-Semidiscretization

- Choose triangulation $\{\mathcal{T}_N\}_N$ s.t. $c|_K \equiv c_j$ for all $K \in \mathcal{T}_N$.
- Fix $p \in \mathbb{N}$. Conforming FEM spaces $V_N := \mathcal{S}^{p,0}(\mathcal{T}_N, \Omega) \cap V$.
- Choose nodal basis functions.

$\Rightarrow$ Solve semidiscrete ODE for $t \in I$:

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{A}\mathbf{u}(t) &= \mathbf{f}(t) , \\ \mathbf{u}(0) &= \mathbf{u}^0 , \\ \dot{\mathbf{u}}(0) &= \mathbf{u}^1 . \end{aligned}$$



Problem description

For $u(\cdot,t) \in H^s(\Omega)$, and on uniform meshes, we have for all $t \in I$:

$$\|u(\cdot,t) - u_N(\cdot,t)\|_{H^1(\Omega)} + \|\partial_t u(\cdot,t) - \partial_t u_N(\cdot,t)\|_{L^2(\Omega)}$$
$$\leq \text{ Error in initial conditions}$$
$$+ \, c\, N^{-(\min(p+1,s)-1)/2} \left[\text{Norms of } \partial_t^j u(\cdot,t),\ j = 0,1,2 \right].$$

Problem: Depending on c_j and the shape of Ω_j, $u(\cdot,t) \in H^{1+\lambda}(\Omega)$ where $\lambda > 0$.

Solution strategy: Singularities are spatially isolated at vertices of Ω_j.
Use locally refined meshes to get rate $N^{-p/2}$.

For stationary equation: Regularity theory and asymptotics well-known.

Outline of the Presentation

Problem description	*[done].*
Regularity	*Find asymptotics towards the vertices.*
Main result	*“Good mesh refinement yields optimal rates...*
Numerical experiments	*...most of the time.”*

Brief overview

- Assume $u^{0,1} \in C_0^\infty(\Omega)$.
  Asymptotics of the IBVP are equivalent to a BVP with transmission conditions [1]

$$\partial_t^2 u(x,t) - \nabla \cdot (c\nabla u(x,t)) = g(x,t)\ , \quad t \in \mathbb{R}\ .$$

- Fourier transform $t \in \mathbb{R} \mapsto \tau \in \mathbb{R} - \mathrm{i}\gamma$ for fixed $\gamma > 0$:
  $\Rightarrow$Parametric elliptic PDE for $\hat{u}(x,\tau)$.
- Apply well-known theory to obtain asymptotics for $\hat{u}(x,\tau)$.
- Backtransform yields asymptotics of $u(x,t)$ *if we can control the coefficients in the asymptotics.*

[1] Kokotov I. Yu. et al.: *Problems of diffraction by a cone: asymptotic behavior of the solutions near the vertex*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 1999

Parametric problem

- Shifted Fourier transform: For $\gamma > 0$ fixed, $\tau \in \mathbb{R} - \mathrm{i}\gamma$,

$$w(t) \mapsto \hat{w}(\tau) := \int_{\mathbb{R}-\mathrm{i}\gamma} e^{-\mathrm{i}\tau t} w(t)\, dt \ ,$$

  satisfies $\widehat{\partial_t^n w}(\tau) = (\mathrm{i}\tau)^n \hat{w}(\tau)$.

- BVP for $\hat{u}(x,\tau)$:

  $-\tau^2 \hat{u}(x,\tau) - \nabla\cdot(c\nabla \hat{u}(x,\tau)) = \hat{g}(x,\tau)$ + same b.c./t.c. as before.

Theorem

For $\hat{g} \in L^2(\Omega)$, $\gamma > 0$, $\sigma \in \mathbb{R}$, $\tau \in \mathbb{R} - \mathrm{i}\gamma$, $\exists!$ solution $\hat{u}$, s.t.

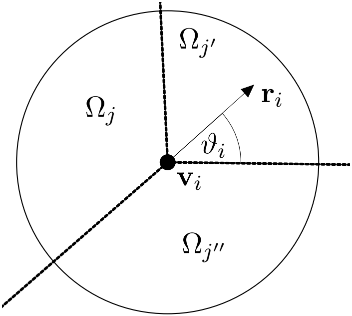
$$\|\hat{u}\|_{H^1}^2 + \|\tau\hat{u}\|_{L^2}^2 \leq c\,\gamma^{-2}\,\|\hat{g}\|_{L^2}^2 \ ,$$

with $c > 0$ independent of $\hat{g}, \hat{u}, \gamma, \tau$.

Localization

- Vertices: $\{\mathbf{v}_i\}_i$: Union of corners of Ω_j: Exterior and interior vertices.
- For all i, choose $R_i > 0$ s.t. $R_i < \frac{1}{2}\min_{i' \neq i} |\mathbf{v}_i - \mathbf{v}_{i'}|$.
- Local domains $\mathcal{D}_i := \Omega \cap B_R(\mathbf{v}_i)$.
- Polar coordinates: (r_i, ϑ_i) centered at $\mathbf{v}_i$.
- Cut-off: $\chi_i \in C^\infty(\bar{\Omega})$ decreasing, s.t.

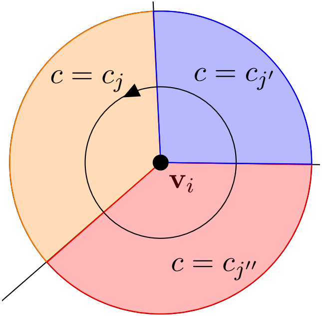
$$\chi_i(x) = \begin{cases} 1 & |x - \mathbf{v}_i| < \frac{R_i}{2} \\ 0 & x \in \bar{\Omega} \setminus \bar{\Omega}_i \end{cases}$$



Singular functions I

- Operator in (r, ϑ):
  $\mathcal{L}_k(\vartheta, \partial_\vartheta, r\partial_r) := -\frac{c_k}{r}\partial_r(r\partial_r) - \frac{c_k}{r^2}\partial_\vartheta^2$.
- Homogeneity of order 2:
  $\mathcal{L}_k[r^\lambda \Theta(\vartheta)] = r^{\lambda-2}\mathcal{L}_k[\Theta(\vartheta)]$
- Change of variable
  $\mathbb{R}_{\geq 0} \ni r \mapsto \rho := -\log r \in \mathbb{R}$.
- Fourier transform $\rho \mapsto \lambda \in \mathbb{C}$
  $\Rightarrow$ Parametric problem for $\vartheta \in [0, 2\pi]$.

  Solve a Sturm-Liouville problem:
  Find (λ, ψ), s.t. $\mathfrak{A}(\lambda)[\psi(\vartheta)] = 0$, where
  $\mathfrak{A}(\lambda)[w] := \left\{r^{2-\lambda}\,\mathcal{L}_k(r^\lambda w(r, \vartheta))\right\}_k$
  + transmission and boundary conditions.



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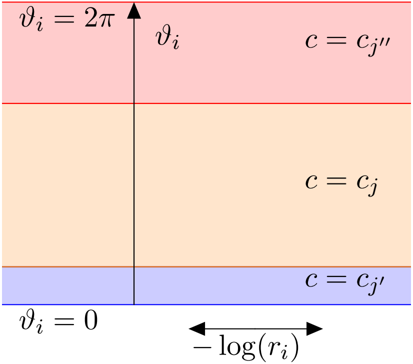


Singular Functions II

- Ellipticity of $\mathcal{L}$: Eigenvalues $\{\lambda_n\} > 0$, isolated. [2]
- Singular functions associated to $\mathbf{v}_i$: Eigenvalue λ and $\{\psi^i_{\lambda,\mu,k}\}_{\mu,k}$ system of Jordan chains.

$$v_{\lambda,\mu,k}(x) := r_i(x)^\lambda \sum_{l=0}^{k} \frac{1}{l!} \log(r_i(x))^l \psi^i_{\lambda,\mu,k-l}(\vartheta_i(x))$$

(μ, k run over finite set of integer indices.)
- $\psi^i_{\lambda,\mu,l}$ are smooth in ϑ_i.
- Only know $\lambda > 0$.
$\Rightarrow v_{\lambda,\mu,k}(x) \in H^1(\mathcal{D}_i)$ is maximal regularity in terms of classical Sobolev spaces!

[2] Nicaise and Sändig 1994,

Kozlov, Maz'ya, Rossmann: *Spectral problems associated with corner singularities of solutions to elliptic equations.* 2001.

Weighted Sobolev spaces

- Suitable scale for functions $x \mapsto r_i^{\lambda} \log(r_i)^l \psi(\vartheta)$.
- Weight $\beta \in \mathbb{R}$, regularity index $s \in \mathbb{N}_0$. Space H^s_β with norm

$$\|w; H^s_\beta(\Omega_j)\| := \sqrt{\sum_{|\alpha|<s} \left\| \prod_i r_i^{\beta-s+|\alpha|} \mathtt{D}^\alpha w \right\|_{L^2}^2} \,.$$

- Total space: $H^s_\beta(\Omega) := \left\{ w \in L^2(\Omega) \,|\, w|_{\Omega_j} \equiv w_j \in H^s_\beta(\Omega_j) \right\}$.

Asymptotics for the parametric problem

Theorem (Nicaise and Sändig 1994)

Let $\hat{g} \in H^s_\beta(\Omega)$ with $\beta \leq s$ (except finitely many values).
$\exists!$ solution $\hat{u}(x,\tau)$ to the parametric problem satisfying

$$\hat{u}(x,\tau) = \hat{u}_0(x,\tau) + \sum_i \chi_i(x) \sum_{\lambda,\mu,k:\ \text{finite}} \hat{c}_{i,\lambda,\mu,k}(\tau)\, v^i_{\lambda,\mu,k}(x),$$

where

$$\|\hat{u}_0(\cdot,\tau); H^{s+2}(\Omega)\| + \sum |\hat{c}_{i,\lambda,\mu,k}(\tau)| \leq C(\tau)\|\hat{g}; H^s_\beta(\Omega)\| \ .$$

Combined estimates for Wave equation I

Recall $\tau \in \mathbb{R} - \mathrm{i}\gamma$ for $\gamma > 0$ fixed.

"Time-dependent" weighted Sobolev spaces:

$$\|\hat{w}; H^s_\beta(\Omega, \tau)\| := \sqrt{\sum_{k \leq s} \|\tau^k \hat{w}; H^{s-k}_\beta(\Omega)\|^2}\ ,$$

$$\|\hat{u}; \mathrm{DH}_{\beta,s}(\Omega, \tau)\| := \sqrt{\gamma^2 \|\hat{u}; H^{s+1}_{\beta+s}(\Omega, \tau))\|^2 + \sum_i \|\chi^i(|\tau|\, x)\hat{u}; H^{s+2}_{\beta+s}(\Omega; \tau)\|^2}\ ,$$

$$\|\hat{g}; \mathrm{RH}_{\beta,s}(\Omega, \tau)\| := \sqrt{\|\hat{g}; H^s_{\beta+s}(\Omega, \tau)\|^2 + \frac{1}{\gamma^{1+s}} \|\tau^{1-\beta+s}\hat{g}\|^2_{L^2(\Omega)}}\ .$$

Combined estimates for Wave equation II

Theorem

Let $\beta \leq 1$ as in the Theorem before, and let $\gamma > 0$ fixed.
Let $\hat{u} \in \mathsf{DH}_{\beta,s}(\Omega,\tau)$ and let $\hat{g} := -\tau^2\hat{u} - \nabla \cdot (c\nabla(\hat{u}))$. There is a constant $C > 0$, dependent on β, s, but ***independent of*** *$\tau, \hat{g}, \hat{u}$, s.t.*

$$\|\hat{u}(\cdot,\tau); \mathsf{DH}_{\beta,s}(\Omega,\tau)\|^2 \leq C\|\hat{g}(\cdot,\tau); \mathsf{RH}_{\beta,s}(\Omega,\tau)\| \ .$$

- Strategy first used by Plamenevskiĭ 1998 and Plamenevskiĭ et al. 2000-2006.
- Lose one smoothness index of $\hat{u}$ to χ-term.
- Require more smoothness on g in t.

Combined estimates for Wave equation III

Proof sketch:

- Start with $s = 0$.
- Consider $\mathbb{R}^2$ (or a conical subset) as neighborhood of the origin.
- Estimate for $\chi\hat{u}$ near the origin: already done.
- Estimate in neighborhoods of ∞:

$$\gamma^2 \|\hat{u}; H^1_\beta(\mathbb{R}^2, \tau)\|^2 \;\leq\; c\,\|\kappa_\infty(|\tau|x)\,\hat{g}; H^1_{\beta-1}(\mathbb{R}^2; \tau)\|\ .$$

- Add inequalities with $\mathrm{supp}(\kappa_\infty) \subseteq \mathrm{supp}(1-\chi)$.
- Induction step to conclude for $s > 0$.

Regularity result for the Wave equation

Theorem

Assume $u^0, u^1 \in C_0^\infty(\Omega)$ *and* $f \in C^0(\bar{I}; H^p(\Omega))$.
Then $\exists!$ *solution* $u(x,t) \in C^2(\bar{I}; V^*)$ *to the initial-boundary-value problem which allows* $\forall p \in \mathbb{N}$ *the following decomposition:*

$$u(x,t) = u_0(x,t) + \sum_i \chi_i(x) \sum_{\lambda,\mu,k:\ \text{finite}} c^i_{\lambda,\mu,k}(t)\, v^i_{\lambda,\mu,k}(x)\ ,$$

with $u_0 \in C^0(\bar{I}; H^{p+1}(\Omega))$ *and* $c^i_{\lambda,\mu,k} \in C^2(\bar{I})$.

- $u(x,t)$ obtained by backtransform of $\hat{u}(x,\tau)$. Uniform bounds (combined estimate): well-defined asymptotics for $u(x,t)$ and bounded $c^i_{\lambda,\mu,k}$.
- Under the assumptions on $u^{0,1}, f$, the equivalent reformulation as a BVP is possible.

Local mesh refinement

Problem: Fixed $t = t_0$, $\lambda := \min_i \lambda_{i,\min} > 0$.
Quasi-uniform $(\mathcal{T}_N)_N$ approximate $u(\cdot, t_0)$ with rate $\mathcal{O}(N^{-\frac{\varrho}{2}})$ with $\varrho < \min(p, \lambda)$ on $S^{p,0}(\Omega, \mathcal{T}_N)$.
Generally, $\lambda > 0$ is only bound.

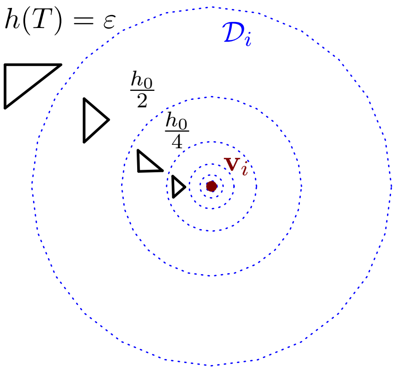
Idea: Locally refined mesh families $(\mathcal{T}_N)_N$ are known to approximate $u(\cdot, t_0)$ with optimal convergence rate $\mathcal{O}(N^{-\frac{p}{2}})$.

M. and Schwab: *Finite Elements with mesh refinement for wave equations in polygons*, SAM-Report 2013.11

Local bisection refinement towards $\mathbf{v}_i$

Given regular initial triangulation $\mathcal{T}_0$, parameter $\varepsilon > 0$.

1. Refine until $h \leq \varepsilon$ everywhere.
2. `For` $l = 1 : K$
   bisect all elements T, with $\mathrm{dist}(\mathbf{v}_i, T) \leq 2^{-l} R_i$, and $h(T) > h_{0,l}$.



Refinement number K depends on

- $\lambda = \min_i \lambda_{i,\min}$,
- FEM degree $p \in \mathbb{N}$.

If $\lambda \to 0$ or $p \to \infty$, $K \to \infty$.

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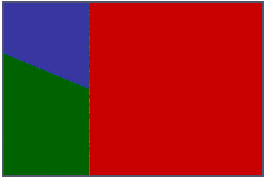

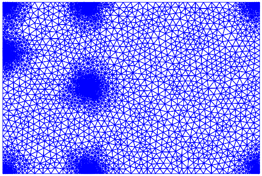

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Main result

Theorem

Given $p \in \mathbb{N}$ *and assume* $u^{0,1} \in C_0^\infty(\Omega)$, $f \in C^0(\bar{I}; H^p(\Omega))$.

Let $\{\mathcal{T}_N\}_N$ *be a family of meshes s.t.*

$$\inf_{v \in S^{p,0}(\Omega,\mathcal{T}_N)} \|v^i_{\lambda,\mu,k} - v\|_{H^1} = \mathcal{O}(N^{-p/2})$$

$$\text{and} \quad \inf_{v \in S^{p,0}(\Omega,\mathcal{T}_N)} \|w - v\|_{H^1} = \mathcal{O}(N^{-p/2}) \quad \forall\, w \in H^{p+1}(\Omega)\ .$$

Let $u_N(\cdot,t) \in S^{p,0}(\Omega,\mathcal{T}_N)$ *be the semi-discrete FEM solutions. There holds for all* $t \in I$*:*

$$\begin{aligned} \|u(\cdot,t) - u_N(\cdot,t)\|_{H^1} + \|\partial_t u(\cdot,t) - \partial_t u_N(\cdot,t)\|_{L^2} \\ \leq \ & \text{Error in initial conditions} \\ & + c\,N^{-p/2}\left[\text{Norms of } \partial_t^j u(\cdot,t),\ j = 0,1,2\right]. \end{aligned}$$

Proof sketch

- At fixed time t, apply decomposition into singular and nonsingular parts:

$$u = u_0 + \sum_i \chi_i u_{\text{singular},i} \ .$$

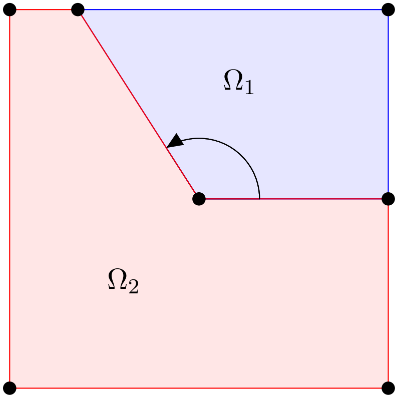
- By the decomposition theorem: $u_0(\cdot, t) \in H^{p+1}(\Omega)$.
- Singular terms get approximated with optimal convergence rate.
  Regular term u_0 gets approximated with optimal convergence rate.
  $\Rightarrow u(x, t)$ gets approximated with optimal convergence rate.

Remarks

- Consider also $\mathbf{v}_i \in \partial\Omega$.
- In 2D, edges between interfaces do not yield singularities.
- Singularities spread along Ω with finite propagation speed. The “emissions” of $\{\mathbf{v}_i\}_i$ are nonsingular away from $\mathbf{v}_i$.

Test 1: Bi-material

- $\Omega = (-1,1)^2$, Dirichlet conditions,
  $\Omega_1 := \{(r,\vartheta) \,|\, \vartheta \in (0,\omega_1)\}$.
- $u^{0,1} \equiv 0$,
  $f(x,t) = C^\infty$ bump in $(0,0)$.
- $c_1 = .25$, $c_2 = 1.5$.
- For $\omega_1 = \frac{\pi}{4}$: $\lambda \simeq 0.4$.
- $T = 1$ for linear FEM and 0.5 for quadratic FEM.
- Implicit time-stepping with $\Delta t = 10^{-4}$ for linear FEM and $\Delta t = 10^{-5}$ for quadratic FEM.



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  $\Omega_1 := \{(r,\vartheta) \,|\, \vartheta \in (0,\omega_1)\}$.
- $u^{0,1} \equiv 0$,
  $f(x,t) =$hat function in Ω_1,
  $f((0,0),t) = 1$.
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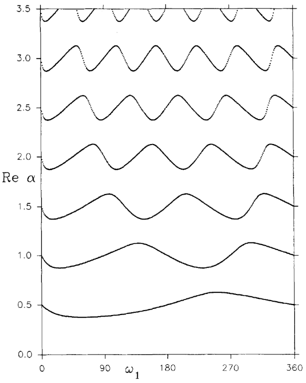
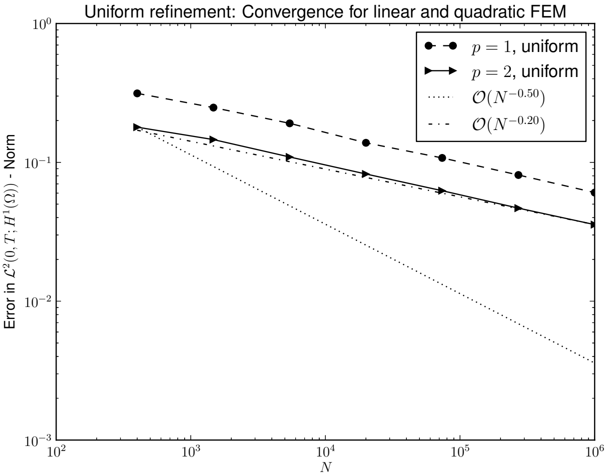
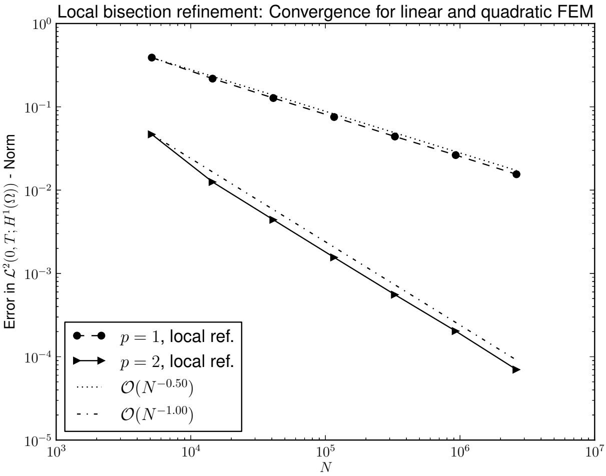


Source: Nicaise and Sändig 1994

Test 1: Uniform refinement

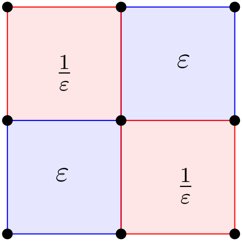


Test 1: Local bisection refinement



Test 2: Problem with almost vanishing wavespeed

- $\Omega = (-1,1)^2$ subdivided into four quadrants.
- $c_1 = c_3 = \varepsilon > 0$, $c_2 = c_4 = \frac{1}{\varepsilon}$.
- Implicit formulae for λ: Kellogg, 1975
- If $\varepsilon \searrow 0$, $\lambda \searrow 0$.
- Asymptotically, convergence rate still optimal.
- More local refinement levels are needed.
- nonphysical.



Kellogg, R.B.: *On the Poisson Equation with Intersecting Interfaces*, Appl. Anal., 1975.

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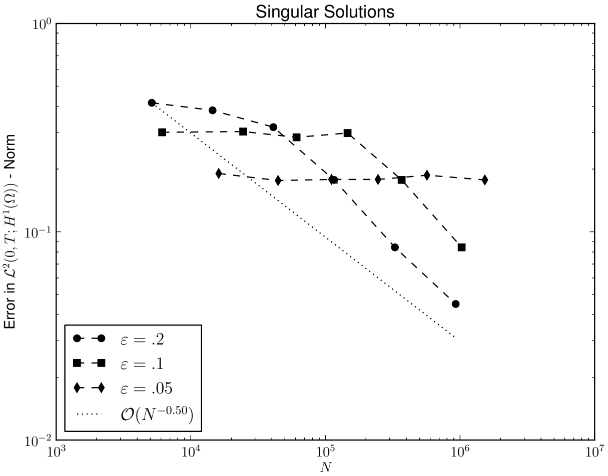

Limitations

- Norm $\|u(\cdot,t)\|_{H^1(\Omega)}$ in the error estimate $\to \infty$ as $\lambda \to 0$.
- $\lambda \searrow 0$, #refinements$\to \infty \;\Rightarrow \min_{T\in\mathcal{T}_N} h_T \to 0$.

  In the error estimate $\inf_{v\in S^{p,0}(\Omega,\mathcal{T}_N)} \|v^i_{\lambda,\mu,k} - v\|_{H^1} \leq cN^{-p/2}$, $c \to \infty$ if $K \to \infty$.

$\Rightarrow$ If $\lambda \searrow 0$, the constants are too large to ensure convergence within reasonable time.

Conclusion

- The piecewise constant Transmission problem for the Wave equation has a unique solution in $H^1(\Omega)$ with singularities.
  $\Rightarrow$ Thus, FEM on uniform mesh families converges slowly.
- The singularities appear in a decomposition isolated at the points of nonsmoothness.
- The singular terms lie in weighted Sobolev spaces, hence locally refined meshes yield optimal convergence rates.
- Possible constellations of the data yield diverging constants in the error estimate: Bad convergence.

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Thank you for your attention