Numerical Approximation of Elliptic Equations with Point-Source Forcing Term

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Elliptic PDEs with Point Source Forcing Term

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. We consider the following problem: find $u \in X$ such that

$$\begin{aligned} -\Delta u &= \delta_x \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where δ_x is the Dirac measure at some given $x \in \Omega$. For simplicity, we assume that $x = 0 \in \Omega$ and we write $\delta_x = \delta$.
The weak formulation of (1) is then: find $u_\delta \in X$ such that

$$a(u_\delta, v) := \int_\Omega \nabla u_\delta \cdot \nabla v = \langle \delta, v \rangle, \qquad \forall v \in Y, \tag{2}$$

where $\langle \cdot, \cdot \rangle$ denotes the $Y' - Y$ duality pairing.
Problem:

- Sobolev embedding theorem [1] implies $W_0^{1,q}(\Omega) \subset C(\Omega)$ only for $2 < q < \infty$;
- Need a theory for $W_0^{1,p}(\Omega) - W_0^{1,q}(\Omega)$ spaces $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$ instead of the usual $H_0^1(\Omega)$ one.

An Abstract Result [2]

Theorem

Let X and Y be Banach spaces such that Y is reflexive, $F \in Y'$ and $a(\cdot,\cdot) : X \times Y \to \mathbb{R}$ be a bilinear form such that

(i) *(continuity) there exists $\gamma > 0$ such that*

$$a(u,v) \leq \gamma \|u\|_X \|v\|_Y, \;\; u \in X, v \in Y;$$

(ii) *(inf-sup) there exists $\alpha > 0$ such that*

$$\sup_{v \in Y} \tfrac{a(u,v)}{\|v\|_Y} \geq \alpha \|u\|_X, \;\; u \in X,$$
$$\sup_{u \in X} \tfrac{a(u,v)}{\|u\|_X} \geq \alpha \|v\|_Y, \;\; v \in Y.$$

Then there exists a unique $u \in X$ satisfying

$$a(u,v) = F(v), \;\; \forall v \in Y. \tag{3}$$

Moreover, it holds

$$\|u\|_X \leq \frac{1}{\alpha} \|F\|_{Y'}. \tag{4}$$

Star-Shaped Supported Function

For some given parameter $\varepsilon > 0$, let us consider the star-shaped domain

$$D_\varepsilon := \{(r\cos(\theta), r\sin(\theta)) \mid \theta \in [0, 2\pi],\ r \in [0, r_\varepsilon(\theta)]\}\,, \tag{5}$$

and we assume that there exists $c, C > 0$ such that

$$c\varepsilon \leq r_\varepsilon(\theta) \leq C\varepsilon, \qquad \forall\theta \in [0, 2\pi],\ \forall\varepsilon > 0.$$

We then define the function

$$f_\varepsilon := \frac{1}{\mu(D_\varepsilon)}\chi_{D_\varepsilon}\,, \tag{6}$$

where χ_{D_ε} is the characteristic function over D_ε and $\mu(D_\varepsilon)$ its measure. We have $f_\varepsilon \in L^\infty(\Omega)$ and so $f_\varepsilon \in W^{-1,p}(\Omega) := \left(W_0^{1,q}(\Omega)\right)'$ for $1 < p < \infty$.

Proposition

It holds

$$f_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} \delta \text{ in } \mathcal{D}'(\Omega).$$

We then consider the auxiliary problem: find $u_\varepsilon \in W_0^{1,p}(\Omega)$ such that

$$a(u_\varepsilon, v) = \langle f_\varepsilon, v\rangle, \qquad \forall v \in W_0^{1,q}(\Omega). \tag{7}$$

Idea:

- Since $f_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \delta$, we want to approximate u_δ by u_ε;
- If $\|u_\varepsilon\|_{H_0^1(\Omega)}$ blows up gently as $\varepsilon \to 0$, we can obtain reasonible approximations without too many computational efforts.

Well-Posedness:

- $X = W_0^{1,p}(\Omega)$, $Y = W_0^{1,q}(\Omega)$;
- $a(u, v) = \int_\Omega \nabla u \cdot \nabla v$
  1. The continuity follows from Hölder's inequality;
  2. The inf-sup condition has been proven by Verfürth in [3] with a constant α_p;
- Right-hand side
  1. $f_\varepsilon \in L^p(\Omega) \subset W^{-1,p}(\Omega)$ for $1 < p < \infty$;
  2. $\delta \in W^{-1,p}(\Omega)$ for $1 < p < 2$.

Behavior of $\|u_\varepsilon\|_{H_0^1(\Omega)}$ for a Circle

For a given $R > 0$, let us first assume that $D_\varepsilon = B_\varepsilon(0)$ and $\Omega = B_R(0)$. In that case we have

$$u_\varepsilon(r,\theta) := -\frac{1}{4\pi}\left(\frac{r^2}{\varepsilon^2} + 2\log\left(\frac{\varepsilon}{R}\right) - 1\right)\chi_{B_\varepsilon(0)} - \frac{1}{2\pi}\log\left(\frac{r}{R}\right)\chi_{B_R(0)\setminus B_\varepsilon(0)},$$

and for $1 < p < 2$ and $\varepsilon > 0$ sufficiently small

$$\|\nabla u_\varepsilon\|_{L^2(\Omega)} \leq \sqrt{\frac{|\log(\varepsilon)|}{\pi}},$$

$$\|\nabla u_\varepsilon\|_{L^p(\Omega)} = \frac{1}{(2\pi)^{1/q}(2-p)^{1/p}}\left(R^{2-p} - \frac{2p\varepsilon^{2-p}}{p+2}\right)^{1/p}.$$

Behavior of $\|u_\varepsilon\|_{H_0^1(\Omega)}$ for Star-Shaped Domains

Lemma

For $1 < p < \infty$, *let* $f, g \in W^{-1,p}(\Omega)$ *be such that there exists* $B > 0$ *with* $\|f\|_{W^{-1,p}(\Omega)} \leq B\|g\|_{W^{-1,p}(\Omega)}$. *Their associated solutions* u_f *and* u_g *then satisfy*

$$\|u_f\|_{W_0^{1,p}(\Omega)} \leq \frac{B}{\alpha_p}\|u_g\|_{W_0^{1,p}(\Omega)}.$$

Remark: Due to the assumption $c\varepsilon \leq r_\varepsilon(\theta) \leq C\varepsilon$, we have

$$\|f_\varepsilon\|_{W^{-1,p}(\Omega)} \leq \left(\frac{C}{c}\right)^2 \|g_{C\varepsilon}\|_{W^{-1,p}(\Omega)},$$

where $g_{C\varepsilon} := \frac{1}{\mu(B_{C\varepsilon}(0))}\chi_{B_{C\varepsilon}(0)}$.

Behavior of $\|u_\varepsilon\|_{H_0^1(\Omega)}$ for Star-Shaped Domains

For a given domain $\bar{\Omega} \subset \tilde{\Omega}$ and $u \in W_0^{1,p}(\Omega)$, we define its zero extension $\tilde{u}$ to $\tilde{\Omega}$ as

$$\tilde{u} := \begin{cases} u & \text{in } \Omega, \\ 0 & \text{in } \tilde{\Omega} \setminus \Omega. \end{cases}$$

Then $\tilde{u} \in W_0^{1,p}(\tilde{\Omega})$ and there exists a constant $C_p > 0$ such that

$$\|u\|_{W_0^{1,p}(\Omega)} \leq \|\tilde{u}\|_{W_0^{1,p}(\tilde{\Omega})} \leq C_p \|u\|_{W_0^{1,p}(\Omega)}. \tag{8}$$

Lemma

Let $f \in L^p(\Omega)$ satisfy $\operatorname{supp}\{f\} \subsetneq \Omega$ and consider $\tilde{\Omega}$ such that $\Omega \subset \bar{\Omega} \subset \tilde{\Omega}$. Then

$$\|f\|_{W^{-1,p}(\Omega)} \leq \frac{1}{C_p} \|\tilde{f}\|_{W^{-1,p}(\tilde{\Omega})}.$$

Remark: Since Ω is a bounded polygonal domain, there exists R such that $\Omega \subset B_R(0)$.

Behavior of $\|u_\varepsilon\|_{H_0^1(\Omega)}$ for Star-Shaped Domains

Proposition

Let Ω be a bounded polygonal domain such that $\Omega \subset B_R(0)$ for some $R > 0$. For $\varepsilon > 0$ sufficiently small, let f_ε be defined as (6) for a star shaped domain D_ε and assume that there exists constants c and C such that $c\varepsilon \le r_\varepsilon(\theta) \le C\varepsilon$ holds. Then, for $1 < p \le 2$, there exists a constant $\mathcal{C}_p$ independent of ε such that the solution u_ε of (2) satisfies

$$\begin{aligned} \|u_\varepsilon\|_{H_0^1(\Omega)} &\le \mathcal{C}_2 \sqrt{|\log(\varepsilon)|}, \\ \|u_\varepsilon\|_{W_0^{1,p}(\Omega)} &\le \mathcal{C}_p \left(\left(\frac{R}{C}\right)^{2-p} - \frac{2p\varepsilon^{2-p}}{p+2} \right)^{1/p}. \end{aligned}$$

Remarks:

- As desired, the blow-up of $\|u_\varepsilon\|_{H_0^1(\Omega)}$ is gentle;
- $\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{W_0^{1,p}(\Omega)} < \infty$ for $1 < p < 2$;
- $\lim_{p\to 2} \|u_\varepsilon\|_{W_0^{1,p}(\Omega)} = \|u_\varepsilon\|_{H_0^1(\Omega)}$ for $\varepsilon > 0$.

Finite Element Approximation

For a shape-regular mesh $\mathcal{T}$ of Ω, we define

$$S^1(\mathcal{T}) := \left\{ \varphi \in C(\Omega) \;\middle|\; \varphi|_T \in \mathbb{P}^1(T),\ \forall T \in \mathcal{T} \right\}.$$

The FE problem then becomes: find $u_{\varepsilon,\mathcal{T}} \in S^1(\mathcal{T})$ such that

$$a(u_{\varepsilon,\mathcal{T}}, v_{\mathcal{T}}) = \langle f_\varepsilon, v_{\mathcal{T}} \rangle,\ \forall v_{\mathcal{T}} \in S^1(\mathcal{T}). \tag{9}$$

For the well-posedness of (9), we use the same theorem as in the continuous case. In particular, the following inf-sup condition holds [3]

$$\sup_{w_{\mathcal{T}} \in S^1(\mathcal{T})} \frac{\int_\Omega \nabla v_{\mathcal{T}} \cdot \nabla w_{\mathcal{T}}}{\|w_{\mathcal{T}}\|_{W_0^{1,q}(\Omega)}} \geq \beta_p \|v_{\mathcal{T}}\|_{W_0^{1,p}(\Omega)}.$$

Moreover, we have the following best approximation property

$$\|u_\varepsilon - u_{\varepsilon,\mathcal{T}}\|_{W_0^{1,p}(\Omega)} \leq \left(1 + \frac{1}{\beta_p}\right) \inf_{v_{\mathcal{T}} \in S^1(\mathcal{T})} \|u_\varepsilon - v_{\mathcal{T}}\|_{W_0^{1,p}(\Omega)}.$$

Convergence of the Finite Element Approximation

Let $h := \max\{\operatorname{diam}(T) \mid T \in \mathcal{T}\}$. The following convergence result for the FE approximation holds.

Theorem

Let $\mathcal{T}$ be an admissible shape regular mesh. Then there exists a constant $C > 0$ such that for $u \in W_0^{m,p}(\Omega)$ with $1 < p \le 2$ and $m = 1$ or 2, it holds

$$\|u - u_{\mathcal{T}}\|_{W_0^{l,p}(\Omega)} \le C h^{k-l} \|u\|_{W_0^{k,p}(\Omega)}, \qquad l = 0, \ldots, k,$$

with the convention that $\|u - u_{\mathcal{T}}\|_{W_0^{0,p}(\Omega)} = \|u - u_{\mathcal{T}}\|_{L^p(\Omega)}$.

In our particular case

$$\begin{aligned} \|u_\varepsilon - u_{\varepsilon,\mathcal{T}}\|_{L^p(\Omega)} &\le Ch \left(\left(\frac{R}{C}\right)^{2-p} - \frac{2p\varepsilon^{2-p}}{p+2} \right)^{1/p}, \\ \|u_\varepsilon - u_{\varepsilon,\mathcal{T}}\|_{L^2(\Omega)} &\le Ch\sqrt{|\log(\varepsilon)|}. \end{aligned}$$

Numerical Results

Let us consider $D_\varepsilon = B_\varepsilon(0)$ and $\Omega = B_1(0)$. We know that the solution is

$$u_\varepsilon(r,\theta) := -\frac{1}{4\pi}\left(\frac{r^2}{\varepsilon^2} + 2\log(\varepsilon) - 1\right)\chi_{B_\varepsilon(0)} - \frac{1}{2\pi}\log(r)\,\chi_{B_1(0)\setminus B_\varepsilon(0)}$$

In that case for $1 < p \leq 2$ and small enough $\varepsilon > 0$

$$\|u_\varepsilon\|_{H_0^1(\Omega)} \leq C_2\sqrt{|\log(\varepsilon)|}, \quad \|u_\varepsilon\|_{W_0^{1,p}(\Omega)} \leq \frac{C_p}{(2-p)^{1/p}}\left(\left(\frac{1}{C}\right)^{2-p} - \frac{2p\varepsilon^{2-p}}{p+2}\right)^{1/p},$$

$$\|u_\varepsilon\|_{H_0^2(\Omega)} \leq \frac{C_2}{\varepsilon}, \quad \|u_\varepsilon\|_{W_0^{2,p}(\Omega)} \leq \frac{C_p}{(2-2p)^{1/p}}\varepsilon^{\frac{2-2p}{p}}.$$

It then follows

$$\|u_\varepsilon - u_{\varepsilon,\mathcal{T}}\|_{H_0^1(\Omega)} \leq C\min\left\{\sqrt{|\log(\varepsilon)|}, \frac{h}{\varepsilon}\right\},$$

$$\|u_\varepsilon - u_{\varepsilon,\mathcal{T}}\|_{W_0^{1,p}(\Omega)} \leq C\min\left\{\frac{1}{(2-p)^{1/p}}\left(\frac{1}{C^{2-p}} - \frac{2p\varepsilon^{2-p}}{p+2}\right)^{1/p}, h\left(\frac{\varepsilon^{2-2p}}{2-2p}\right)^{1/p}\right\}.$$

All computations have been performed with the C++ library `deal.II` [4].

Solution for $\varepsilon = 0.1$ and $\varepsilon = 0.0001$

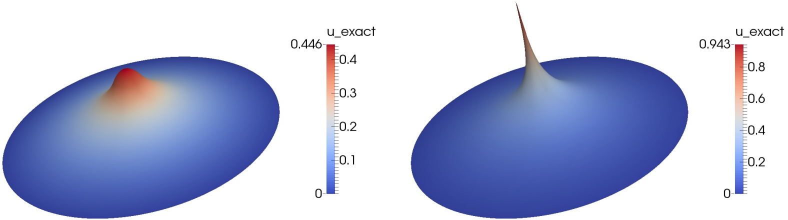
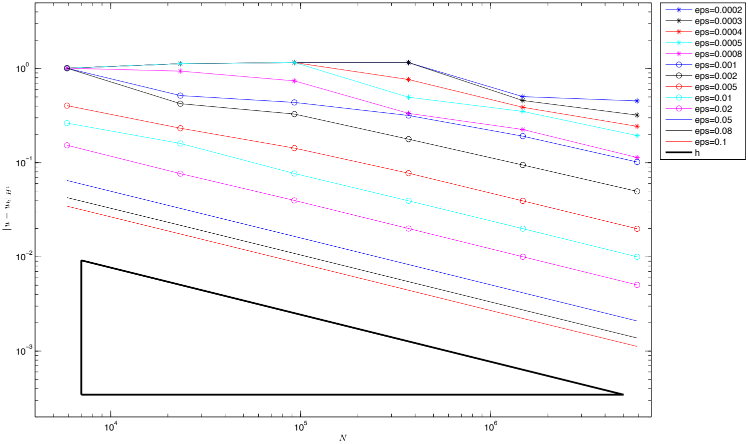
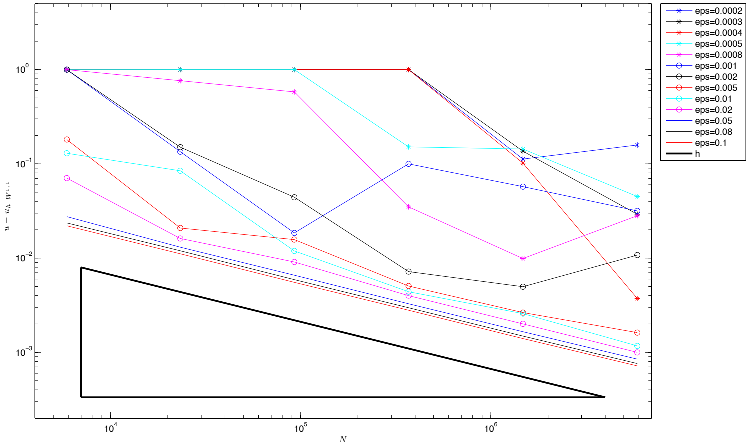


Figure: Exact solution for $\varepsilon = 0.1$ (left) and $\varepsilon = 0.0001$ (right).

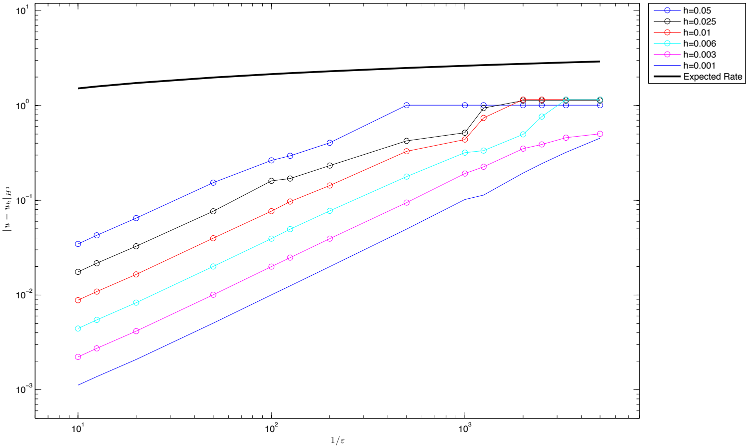
Convergence in the $H_0^1(\Omega)$-norm with respect to h



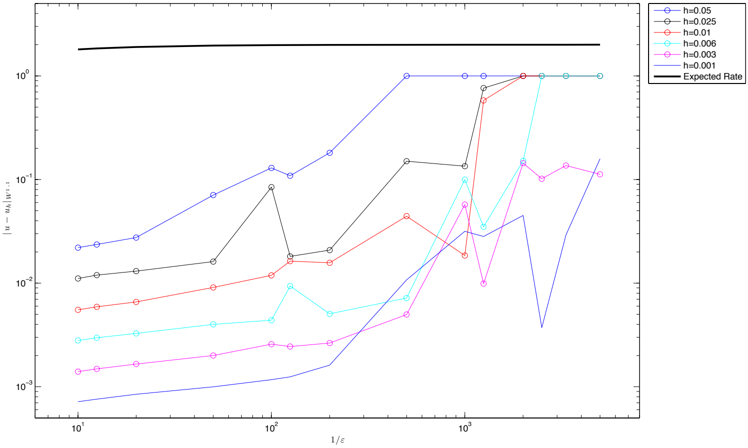
Convergence in the $W_0^{1,1}(\Omega)$-norm with respect to h



Blow-up in the $H_0^1(\Omega)$-norm with respect to ε



Blow-up in the $W_0^{1,1}(\Omega)$-norm with respect to ε



Conclusion

- We want to approximate the solution of the Poisson's equation with a point-source forcing term in a bounded polygonal domain $\Omega \subset \mathbb{R}^2$;
- In this aim, an approximation $f_\varepsilon = \frac{1}{\mu(D_\varepsilon)}\chi_{D_\varepsilon}$ of the Dirac measure has been considered ($B_{c\varepsilon}(0) \subset D_\varepsilon \subset B_{C\varepsilon}(0)$ star-shaped);
- The well-posedness of the elliptic problem in $W_0^{1,p}(\Omega) - W_0^{1,q}(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p < \infty$ has been treated;
- We have introduced a FE approximation considering continuous linear polynomials (with convergence result);
- Some theoretical results concerning the dependence with respect to ε for the constant appearing in the FE convergence have been proposed (giving crucial informations about the preasymptotic phase);
- Numerical results in the particular case of $\Omega = B_1(0)$ and $D_\varepsilon = B_\varepsilon(0)$ have been presented.

Thank you for your attention.

References

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