Numerical Approximation of Parabolic Equations with Singular Data

Denis Devaud[1], Christoph Schwab
ETH Zürich, Seminar for Applied Mathematics

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ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

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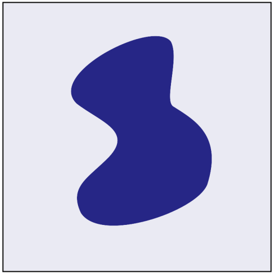
Motivation

- Numerical optimal control;
- Fluid-structure interaction (immersed boundary method [6]);
- $F(\mathbf{x},t) = \int_{D_0(t)} f(\mathbf{s},t)\delta(\mathbf{x}-\mathbf{s})ds$, with $D_0(t) \subsetneq D$ and $D_0(t) \cap \partial D = \emptyset \ \forall t$;



Question: What happens in the case $D_0(t) = \{\mathbf{x}(t)\}$?

Parabolic equations - General setting

- $V \subset H$ Hilbert spaces with dense continuous embedding;
- $D \subset \mathbb{R}^2$ bounded convex polygonal domain;
- self-adjoint operator $A \in \mathcal{L}(V, V^*)$ s.t. the associated bilinear form $a(u, v) = \langle Au, v \rangle$ satisfies

$$a(u, v) \leq \lambda_+ \|u\|_V \|v\|_V, \qquad \forall u, v \in V, \tag{1}$$

$$a(v, v) \geq \lambda_- \|v\|_V^2, \qquad \forall v \in V. \tag{2}$$

- Find $u \in \mathcal{J}_+$ satisfying (in the weak sense)

$$Bu := \partial_t u + Au = f, \qquad \text{in } I_> = (0, T), \tag{3}$$

$$u(0) = 0. \tag{4}$$

In particular: For $X \in C^1(0, T; D)$ and $c \in H^1(0, T)$, we consider

$$f = c(t)\delta_{X(t)},$$

where $\delta_{X(t)}$ is the Dirac measure centered at $X(t)$.

Sobolev spaces

Definition

(i) $H^{1/2}(\mathbb{R}_+;H)$ as the set of functions $u \in L^2(\mathbb{R}_+;H)$ such that

$$\|u\|^2_{H^{1/2}(\mathbb{R}_+;H)} := \|u\|^2_{L^2(\mathbb{R}_+;H)} + \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{\|u(s)-u(t)\|_H^2}{|s-t|^2}\,dsdt < \infty;$$

(ii) $H_0^{1/2}(\mathbb{R}_+;H)$ as the set of functions $u \in H^{1/2}(\mathbb{R}_+;H)$ such that

$$\|u\|^2_{H_0^{1/2}(\mathbb{R}_+;H)} := \|u\|^2_{H^{1/2}(\mathbb{R}_+;H)} + \int_{\mathbb{R}_+} \frac{\|u(s)\|_H^2}{s}\,ds < \infty.$$

Remark: Equivalent to interpolation spaces [5].

Definition

For a subspace $\mathcal{V} \subset V$, we define

$$\mathcal{J}_{+,0} := H_0^{1/2}(\mathbb{R}_+;H) \cap L^2(\mathbb{R}_+;\mathcal{V}),$$
$$\mathcal{J}_{+} := H^{1/2}(\mathbb{R}_+;H) \cap L^2(\mathbb{R}_+;\mathcal{V}).$$

Weak formulation

We define

$$b_+(u,v) := \langle Du, v\rangle + \int_{\mathbb{R}_+} a(u,v)dt, \qquad \forall u \in \mathcal{J}_{+,0},\ v \in \mathcal{J}_+,$$

where (for smooth functions)

$$\langle Du, v\rangle = -\int_{\mathbb{R}_+} (u(t), Dv(t))_H \, dt.$$

Theorem (Fontes [4])

Given $f \in \mathcal{J}_+^*$, there exists a unique $u \in \mathcal{J}_{+,0}$ such that

$$b_+(u,v) = \langle f, v\rangle, \qquad \forall v \in \mathcal{J}_+. \tag{5}$$

Moreover, there exists $C > 0$ such that $\|u\|_{\mathcal{J}_{+,0}} \leq C \|f\|_{\mathcal{J}_+^*}$.

We have $\mathcal{J}_+^* \simeq \left\{ g \in H^{-1/2}(\mathbb{R}; H) + L^2(\mathbb{R}; \mathcal{V}^*) \;\middle|\; \mathrm{supp}\,(g) \subset \mathbb{R}_+ \right\}$.

Note: if $\mathcal{V} = V$, we obtain well-posedness of the original problem.

Increased regularity

For $A \in \mathcal{L}(V, V^*)$, we define

$$D(A) := \{v \in V \mid Av \in H\} .$$

Reminder: We assume that A is a *self-adjoint* operator associated to a coercive and continuous bilinear form.

Theorem

If $f \in H^1(\mathbb{R}_+; H)$, it holds $u \in H^2(\mathbb{R}_+; H) \cap H_0^1(\mathbb{R}_+; D(A))$ and there exists $C > 0$ such that

$$\|u\|_{H^2(\mathbb{R}_+;H)\cap H_0^1(\mathbb{R}_+;D(A))} \leq C \|f\|_{H^1(\mathbb{R}_+;H)} .$$

Proof: Similar to the one presented in Evans [3].

Discretization

- Assume $\mathcal{V} = \mathcal{V}_h$ is *finite dimensional*;
- **Semidiscrete problem** (space approximation): find $u_h \in X_h := H_0^{1/2}(\mathbb{R}_+; H) \cap L^2(\mathbb{R}_+; \mathcal{V}_h)$ such that

$$\langle D u_h, v_h \rangle + \int_{\mathbb{R}_+} a(u_h, v_h) = \langle f, v_h \rangle, \qquad \forall v \in Y_h, \tag{6}$$

  where $Y_h := H^{1/2}(\mathbb{R}_+; H) \cap L^2(\mathbb{R}_+; \mathcal{V}_h)$.
- **Time stepping**: For a time step Δt and $\theta \in [0,1]$, we set $u_0 = 0$ and consider

$$(u_{j+1} - u_j, v_h)_H + \Delta t a(u_j^\theta, v_h) = \Delta t \left\langle f_j^\theta, v_h \right\rangle, \qquad \forall v_h \in \mathcal{V}_h,\ j \in \mathbb{N}, \tag{7}$$

  where $u_j^\theta = \theta u_{j+1} + (1-\theta) u_j$ and $f_j \approx f(j\Delta t)$.

Analysis of the time discretization - Operators

For $\mu \in L^2(\mathbb{R}_+; H)$, we define:

- *Prolongation* operator $\pi^{\mu}_{\Delta t} : \ell^2(H) \to L^2(\mathbb{R}_+; H)$:

$$\pi^{\mu}_{\Delta t}\mathbf{g}(t) := \sum_{i\in\mathbb{N}} g_i \mu\left(\frac{t}{\Delta t} - i\right), \qquad t \in \mathbb{R}_+,\ \mathbf{g} \in \ell^2(H);$$

- *Restriction* operator $\rho^{\mu}_{\Delta t} : L^2(\mathbb{R}_+; H) \to \ell^2(H)$

$$(\rho^{\mu}_{\Delta t} g)_i := \frac{1}{\Delta t}\int_{\mathbb{R}_+} g(t)\mu\left(\frac{t}{\Delta t} - i\right) dt, \qquad i \in \mathbb{N},\ g \in L^2(\mathbb{R}_+; H);$$

- In particular $\pi_0 := \pi^{\psi_0}_{\Delta t}$ and $\pi_1 := \pi^{\psi_1}_{\Delta t}$ (same for ρ) for

$$\psi_0(t) := \chi_{[0,1)}(t),$$
$$\psi_1(t) := \begin{cases} 1+t & \text{if } -1 \le t \le 0, \\ 1-t & \text{if } 0 \le t \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Goal: Obtain stability and convergence results in the $\mathcal{J}_+$-norm.

Analysis of the time discretization - Stability

Proposition

- $\theta \geq 1/2$: *assume* $\|\pi_0 \mathbf{f}\|_{\mathcal{J}_+^*} < \infty$. *Then there exists a constant* $C = C(\lambda_-, \lambda_+) > 0$ *such that*

$$\left\|\pi_1 \mathbf{u}^\theta\right\|_{\mathcal{J}_+} \leq C \|\pi_0 \mathbf{f}\|_{\mathcal{J}_+^*} .$$

  Moreover, if $\theta \neq 1/2$, *there holds*

$$(2\theta - 1) \|\pi_1 \mathbf{u}\|_{\mathcal{J}_{+,0}} \leq C \|\pi_0 \mathbf{f}\|_{\mathcal{J}_+^*} .$$

- $\theta < 1/2$: *assume that* $\|\pi_0 \mathbf{f}\|_{L^2(\mathbb{R}_+;V^*)} < \infty$ *and*

$$\|v_h\|_V \leq C_{inv} h^{-1} \|v_h\|_H , \; \Delta t < \frac{h^2 \lambda_-}{C_{inv}^2 (1 - 2\theta) \lambda_+^2} . \tag{8}$$

  Then there exists $C = C(\lambda_-, \lambda_+) > 0$ *such that*

$$(1 - 2\theta) \|\pi_1 \mathbf{u}\|_{\mathcal{J}_{+,0}} \leq C \|\pi_0 \mathbf{f}\|_{L^2(\mathbb{R}_+;V^*)} .$$

Analysis of the time discretization - Convergence

We define:

$$\mathcal{J}_+^{1,0} := H^2(\mathbb{R}_+; H) \cap H_0^1(\mathbb{R}_+; V),$$
$$\mathcal{S}_+^{1,0} := H^1(\mathbb{R}_+; V^*).$$

Theorem

Let $\theta \in [0,1]$, $f \in \mathcal{J}_+^*$ and set $\mathbf{f} = \rho_1 f$. Assume that $u_h \in \mathcal{J}_+^{1,0}$ and $f \in \mathcal{S}_+^{1,0}$ and if $\theta < 1/2$, assume moreover that the CFL and inverse conditions (8) hold. Then there exists $C > 0$ such that

$$\left\| u_h - \pi_1 \mathbf{u}^\theta \right\|_{\mathcal{J}_+} \leq C \Delta t \left(\|u_h\|_{\mathcal{J}_+^{1,0}} + \|f\|_{\mathcal{S}_+^{1,0}} \right). \tag{9}$$

Moreover, for $\theta \neq 1/2$, there holds

$$|2\theta - 1| \, \|u_h - \pi_1 \mathbf{u}\|_{\mathcal{J}_+} \leq C \Delta t \left(\|u_h\|_{\mathcal{J}_+^{1,0}} + \|f\|_{\mathcal{S}_+^{1,0}} \right). \tag{10}$$

Proof: Based on Baiocchi & Brezzi [1], which prove the previous results on $\mathbb{R}$.

Analysis of the space discretization - A special setting

Setting:

- $Av = -\operatorname{div}(\mathcal{A}(x)\nabla v)$ for a symmetric matrix $\mathcal{A} \in \left(W^{1,\infty}(D)\right)^{2\times 2}$ such that there exists $\lambda_- > 0$ satisfying

$$\sum_{i,j=1}^{2} \mathcal{A}_{i,j}(x)\xi_i\xi_j \geq \lambda_- \left|\xi\right|^2, \qquad \forall x \in D;$$

- $H := L^2(D)$, $V := H_0^1(D)$, where $D \subset \mathbb{R}^2$ is a bounded convex polygonal domain;
- for an *admissible shape regular* mesh $\mathcal{T}_h$:

$$\mathcal{V} := \mathcal{V}_h(\mathcal{T}_h) := \left\{ v \in C^0(D) \;\middle|\; v|_T \in \mathbb{P}^1(T),\ T \in \mathcal{T}_h \right\},$$

  where $\mathbb{P}^1(T)$ is the set of polynomials of degree at most 1 on $T \in \mathcal{T}_h$;

Remark: In that setting, the inverse condition $\|v_h\|_{H_0^1(D)} \leq C_{\text{inv}} h^{-1} \|v_h\|_{L^2(D)}$ holds.

Analysis of the space discretization - A special setting

Regularity:

Lemma

In the previous setting, we have that $D(A) = H^2(D) \cap H_0^1(D)$.

It follows $f \in H^1(\mathbb{R}_+; L^2(D)) \Rightarrow u \in H^2(\mathbb{R}_+; L^2(D)) \cap H_0^1(\mathbb{R}_+; H^2(D) \cap H_0^1(D))$.

Convergence:

We define

$$\mathcal{J}_{+,0}^{0,1} := H_0^{1/2}(\mathbb{R}_+; H_0^1(D)) \cap L^2(\mathbb{R}_+; H_0^1(D) \cap H^2(D)).$$

Theorem

There exists $C > 0$ *such that if* $u \in \mathcal{J}_{+,0}^{0,1}$, *there holds*

$$\|u - u_h\|_{\mathcal{J}_{+,0}} \leq Ch \|u\|_{\mathcal{J}_{+,0}^{0,1}}.$$

Convergence of the space-time discretization

Wrapping up everything, we obtain the main convergence result:

Theorem

Let $\theta \in [0,1]$, $f \in H^1(\mathbb{R}_+; L^2(D))$ and set $\mathbf{f} = \rho_1 f$. If $\theta < 1/2$, assume moreover that the CFL condition (8) holds. Then there exists $C > 0$ such that

$$\left\| u - \pi_1 \mathbf{u}^\theta \right\|_{\mathcal{J}_+} \leq C\,(h + \Delta t)\, \|f\|_{H^1(\mathbb{R}_+;L^2(D))} .$$

Moreover if $\theta \neq 1/2$, there holds

$$\|u - \pi_1 \mathbf{u}\|_{\mathcal{J}_+} \leq \frac{C}{|2\theta - 1|}\,(h + \Delta t)\, \|f\|_{H^1(\mathbb{R}_+;L^2(D))} .$$

Reminder: $\mathcal{J}_+ = H^{1/2}(\mathbb{R}_+; L^2(D)) \cap L^2(\mathbb{R}_+; H^1_0(D))$.
Remark: The ingredients to get this result are:

- The convergence results for the space and time discretizations;
- The regularity result stated above.

Moving point source - Regularization

We consider

$$\begin{aligned} Bu := \partial_t u + Au =& f, \qquad \text{in } I_> = (0, T) \\ u(0) =& 0, \end{aligned}$$

for

$$\boxed{f = c(t)\delta_{X(t)}}.$$

Problem: For any $t \in \mathbb{R}_+$, we have $\delta_{X(t)} \notin H^{-1}(D)$ (Sobolev's embedding).
Idea: Regularize in space: for $\varepsilon > 0$, consider

$$\boxed{f_\varepsilon(t) := \frac{1}{\mu(D_{t,\varepsilon})} c(t)\chi_{D_{t,\varepsilon}}},$$

where $\chi_{D_{t,\varepsilon}}$ is the characteristic function over

$$D_{t,\varepsilon} := \left\{(X_1(t) + r\cos(\theta), X_2(t) + r\sin(\theta)) \mid r \in [0, r_\varepsilon(\theta)],\ \theta \in [0, 2\pi]\right\},$$

assuming $c_1\varepsilon < r_\varepsilon(\theta) < c_2\varepsilon$ for some constants $c_1, c_2 > 0$.

Moving point source - Convergence

We denote:

- u_ε as the solution of (5) for $f = f_\varepsilon$;
- $\mathbf{u}_\varepsilon$ as the solution of (7) for $\mathbf{f} = \rho_1 f_\varepsilon$;

It follows that for $\theta \in [0,1]$ (under the CFL condition for $\theta < 1/2$)

$$\left\| u_\varepsilon - \pi_1 \mathbf{u}_\varepsilon^\theta \right\|_{\mathcal{J}_+} \leq C\,(h + \Delta t)\,\|f\|_{H^1(\mathbb{R}_+;L^2(D))} \approx \frac{C}{\varepsilon}\,(h + \Delta t)\,.$$

If $\theta \neq 1/2$:

$$\| u_\varepsilon - \pi_1 \mathbf{u}_\varepsilon \|_{\mathcal{J}_+} \leq C\,(h + \Delta t)\,\|f\|_{H^1(\mathbb{R}_+;L^2(D))} \approx \frac{C(\theta)}{\varepsilon}\,(h + \Delta t)\,,$$

with $C(\theta) \to \infty$ as $\theta \to 1/2$.

Moving point source - A qualitative example

- $D = [-1,1]^2$, $T = 0.5$;
- $\varepsilon = 10^{-2}$;
- $\Delta t = 5 \cdot 10^{-4}$, $h \approx 8 \cdot 10^{-3}$, $\theta = 1$;

Computations performed with the C++ library `deal.II` [2]

Next steps

- We can show that $\left\| \frac{1}{\mu(D_{t,\varepsilon})}\chi_{D_{t,\varepsilon}} \right\|_{H^{-1}(D)} \approx \sqrt{|\log(\varepsilon)|} \Rightarrow$ derive bounds for the $L^2(\mathbb{R}_+; L^2(D))$-norm with respect to the $H^1(\mathbb{R}_+; H^{-1}(D))$-norm of the right-hand side;
- Computational results;
- Other equations;
- 3D problems;
- Uncertain data.

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Questions?